

“HCR’s Theory of Polygon”

“Solid angle subtended by any polygonal plane at any point in 3D space” 9 Oct, 2014

Mr Harish Chandra Rajpoot

Madan Mohan Malaviya University of Technology, Gorakhpur-273010 (UP) India

Proposed & illustrated by H. C. Rajpoot

I. Introduction

The graphical method overcomes the limitations of all other analytical methods provided the location of foot of perpendicular drawn from the given point to the plane of polygon is known. It involves theoretically zero error if the calculations are done correctly. A polygon is the plane bounded by the straight lines. But, if the given polygon is divided into certain number of elementary triangles then the solid angle subtended by polygon at given point can be determined by summing up the solid angles subtended by all the elementary triangles at the same point. This concept derived a Theory. According to it “for a given configuration of plane & location of point in the space, if a perpendicular, from any given point in the space, is drawn to the plane of given polygon then the polygonal plane (polygon) can be internally or externally or both divided into a certain number of elementary triangles by joining all the vertices of polygon to the foot of perpendicular (F.O.P.) Further each of these triangles can be internally or externally sub-divided in two right triangles having common vertex at the foot of perpendicular. Thus the solid angle subtended by the given polygonal plane at the given point is the algebraic sum of solid angles subtended by all the elementary triangles at the same point such that algebraic sum of the areas of all these triangles is equal to the area of given polygonal plane”

Let’s study in an order to easily understand the Theory of Polygon in a simple way

II. HCR’s Master/Standard Formula-1 (Solid angle subtended by a right triangular plane at any point lying at a normal height h from any of the acute angled vertices)

Using Fundamental Theorem of Solid Angle:

Let there be a right triangular plane ONM having perpendicular ON = p & the base MN = b and a given point say P(0, 0, h) at a height ‘h’ lying on the axis (i.e. Z-axis) passing through the acute angled vertex say ‘O’

(As shown in the figure1 below)

In right $\triangle POR$

$$\Rightarrow \sec\phi = \frac{PR}{PO} \Rightarrow PR = PO \sec\phi = h \sec\phi \text{ \& }$$

$$\tan\phi = \frac{OR}{PO} \Rightarrow OR = PO \tan\phi = h \tan\phi = x$$

Now, the equation of the straight line OM passing through the origin ‘O’

$$\Rightarrow y = mx = \frac{b}{p}x \quad \left(\text{since, slope of line OM} = \frac{b}{p} \right)$$

$$\Rightarrow y = \frac{b}{p}x = \frac{b}{p}h \tan\phi$$

Now, consider a point Q(x, y) on the straight line OM

$$\Rightarrow QR = y = \frac{b}{p} h \tan \varphi$$

In right ΔPRQ

$$\Rightarrow \sin \theta = \frac{QR}{PQ} = \frac{QR}{\sqrt{QR^2 + PR^2}} = \frac{\left(\frac{b}{p} h \tan \varphi\right)}{\sqrt{\left(\frac{b}{p} h \tan \varphi\right)^2 + h^2 \sec^2 \varphi}}$$

$$\sin \theta = \frac{b \sin \varphi}{\sqrt{b^2 \sin^2 \varphi + p^2}}$$

In right ΔPON

$$\Rightarrow \cos \varphi_0 = \frac{OP}{PN} = \frac{OP}{\sqrt{PO^2 + ON^2}} = \frac{h}{\sqrt{h^2 + p^2}}$$

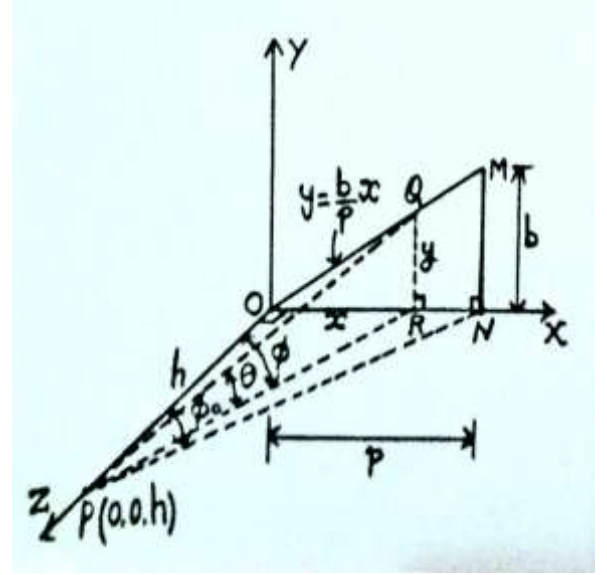


Fig 1: Right Triangular Plane ONM, $ON=p$ & $MN=b$

Let's consider an imaginary spherical surface having radius 'R' & centre at the given point 'P' such that the area of projection of the given plane ONM on the spherical surface w.r.t. the given point 'P' is 'A'

Now, consider an elementary area of projection 'dA' of the plane ONM on the spherical surface in the first quadrant YOZ

(As shown in the figure 2)

Now, elementary area of projection in the first quadrant

$$\Rightarrow dA = (\text{length})(\text{width}) = (R \sin \theta d\varphi)(R d\theta) = R^2 \sin \theta d\theta d\varphi$$

Hence, area of projection of the plane ONM on the spherical surface in the first quadrant is obtained by integrating the above expression in the first quadrant &

Applying the proper limits of ' θ ' from $\theta_1 = (\pi/2 - \theta)$ to $\theta_2 = \pi/2$ & ' φ ' from $\varphi_1 = 0$ to $\varphi_2 = \varphi_0$, we get

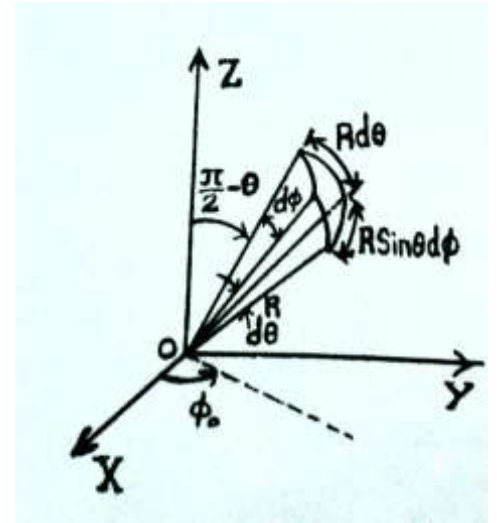


Fig 2: Elementary area dA on spherical surface

$$\Rightarrow A = \int_0^{\varphi_0} \left[\int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}} R^2 \sin \theta d\theta \right] d\varphi = R^2 \int_0^{\varphi_0} \left[\int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}} \sin \theta d\theta \right] d\varphi$$

$$= R^2 \int_0^{\varphi_0} d\varphi [-\cos \theta]_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}} = R^2 \int_0^{\varphi_0} \sin \theta d\varphi = R^2 \int_0^{\varphi_0} \frac{b \sin \varphi}{\sqrt{b^2 \sin^2 \varphi + p^2}} d\varphi$$

$$= R^2 \int_0^{\varphi_0} \frac{b \sin \varphi}{\sqrt{b^2 - b^2 \sin^2 \varphi + p^2}} d\varphi$$

(since, $\sin^2 \alpha = 1 - \cos^2 \alpha$)

$$= R^2 \int_0^{\varphi_0} \frac{b \sin \varphi}{b \sqrt{\frac{b^2 + p^2}{b^2} - \cos^2 \varphi}} d\varphi = R^2 \int_0^{\varphi_0} \frac{\sin \varphi}{\sqrt{K^2 - \cos^2 \varphi}} d\varphi$$

$$\text{where, } K = \sqrt{\frac{b^2 + p^2}{b^2}} = \frac{\sqrt{b^2 + p^2}}{b}$$

$$\begin{aligned} \Rightarrow A &= R^2 \left[-\sin^{-1} \left\{ \frac{\cos \varphi}{K} \right\} \right]_0^{\varphi_0} = R^2 \left[-\sin^{-1} \left\{ \frac{\cos \varphi_0}{K} \right\} + \sin^{-1} \left\{ \frac{\cos 0}{K} \right\} \right] \\ &= R^2 \left[\sin^{-1} \left\{ \frac{1}{K} \right\} - \sin^{-1} \left\{ \frac{\cos \varphi_0}{K} \right\} \right] \end{aligned}$$

Now, on setting the values of 'K' & ' $\cos \varphi_0$ ', we get

$$\begin{aligned} \Rightarrow A &= R^2 \left[\sin^{-1} \left\{ \frac{1}{\frac{\sqrt{b^2 + p^2}}{b}} \right\} - \sin^{-1} \left\{ \frac{\frac{h}{\sqrt{h^2 + p^2}}}{\frac{\sqrt{b^2 + p^2}}{b}} \right\} \right] \\ &= R^2 \left[\sin^{-1} \left\{ \frac{b}{\sqrt{b^2 + p^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{b^2 + p^2}} \right) \left(\frac{h}{\sqrt{h^2 + p^2}} \right) \right\} \right] \end{aligned}$$

Now, the solid angle subtended by the right triangular plane ONM at the given point 'P'

= solid angle subtended by area of projection of triangle ONM on the spherical surface at the same point 'P'

Using Fundamental Theorem

$$\Rightarrow \omega = \int_s \frac{dA}{r^2} = \int_s \frac{dA}{R^2} = \frac{1}{R^2} \int_s dA = \frac{A}{R^2}$$

(since, R is constant for each point on the area of projection on the spherical surface)

$$\begin{aligned} \Rightarrow \omega &= \frac{1}{R^2} \times R^2 \left[\sin^{-1} \left\{ \frac{b}{\sqrt{b^2 + p^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{b^2 + p^2}} \right) \left(\frac{h}{\sqrt{h^2 + p^2}} \right) \right\} \right] \\ &= \sin^{-1} \left\{ \frac{b}{\sqrt{b^2 + p^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{b^2 + p^2}} \right) \left(\frac{h}{\sqrt{h^2 + p^2}} \right) \right\} \\ \Rightarrow \omega &= \left[\sin^{-1} \left\{ \frac{b}{\sqrt{b^2 + p^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{b^2 + p^2}} \right) \left(\frac{h}{\sqrt{h^2 + p^2}} \right) \right\} \right] \dots \dots \dots (1) \end{aligned}$$

If the given point P (0, 0, h) is lying on the axis normal to the plane & passing through the acute angled vertex 'M' then the solid angle subtended by the right triangular plane ONM is obtained by replacing 'b' by 'p' & 'p' by 'b' in the equation(1), we have

$$\Rightarrow \omega = \left[\sin^{-1} \left\{ \frac{p}{\sqrt{p^2 + b^2}} \right\} - \sin^{-1} \left\{ \left(\frac{p}{\sqrt{p^2 + b^2}} \right) \left(\frac{h}{\sqrt{h^2 + b^2}} \right) \right\} \right]$$

Note: Eq(1) is named as **HCR's Master/Standard Formula-1** which is **extremely useful** to find out the solid angle subtended by any polygon at any point in the space thus all the formulae can be derived by using eq(1).

III. Specifying the location of given point & foot of perpendicular in the Plane of Polygon

Let there be any point say point P & any polygonal plane say plane ABCDEF in the space. Draw a perpendicular PO from the point 'P' to the plane of polygon which passes through the point 'O' i.e. **foot of perpendicular (F.O.P.)** (See front & top views in the figure 3 below showing actual location of point 'P' & F.O.P. 'O')

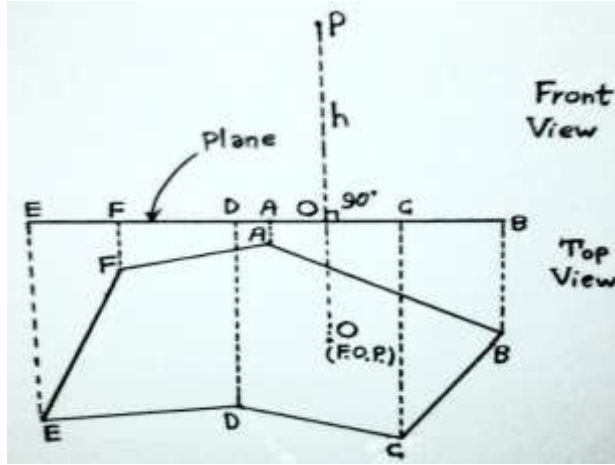


Fig 3: Actual location of point 'P' & F.O.P. 'O'

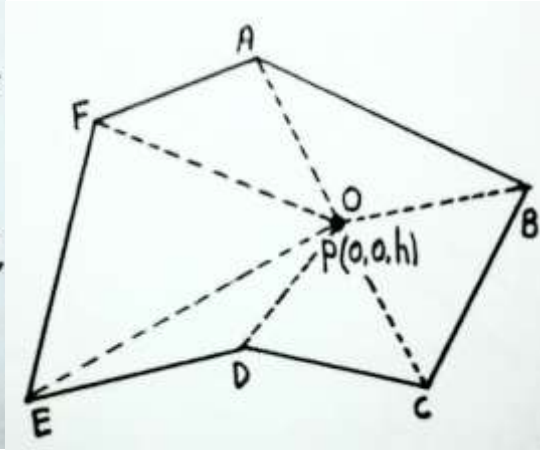


Fig 4: Location of 'P' & 'O' in the plane of paper

Now, for simplification of specifying the location of given point 'P' & foot of perpendicular 'Q' we assume

- Foot of perpendicular 'O' as the origin &
- Point P is lying on the Z-axis

If the length of perpendicular PO is h then the location of given point 'P' & foot of perpendicular 'O' in the plane of polygon (i.e. plane of paper) is denoted by $P(0, 0, h)$ similar to the 3-D co-ordinate system

(See the figure 4 above)

IV. Element Method (Method of dividing the polygon into elementary triangles)

It is the method of Joining all the vertices A, B, C, D, E & F of a given polygon to the foot of perpendicular 'O' drawn from the given point P in the space (as shown by the dotted lines in figure 4 above) Thus, $\Delta AOB, \Delta BOC, \Delta COD, \Delta DOE, \Delta EOF$ & ΔAOF are the **elementary triangles** which have common vertex at the foot of perpendicular 'O'.

Note: $\Delta ABC, \Delta BCD, \Delta CDE, \Delta DEF$ & ΔEFA are not taken as the elementary triangles since they don't have common vertex at the foot of perpendicular 'O'

The area (A_{polygon}) of polygon ABCDEF is given as follows (from the figure 4)

$$A_{\text{polygon}} = \text{algebraic sum of areas of elementary triangles} \\ = A_{\Delta AOB} + A_{\Delta BOC} + A_{\Delta COD} + A_{\Delta DOE} + A_{\Delta EOF} + A_{\Delta AOF}$$

Since the **location of given point & the configuration of polygonal plane is not changed** hence the solid angle (ω_{polygon}) subtended by polygonal plane at the given point in the space is the algebraic sum of solid angles subtended by the elementary triangles obtained by joining all the vertices of polygon to the F.O.P.

Now, replacing the areas of elementary triangles by their respective values of solid angles in the above expression as follows

$\omega_{polygon} = \text{algebraic sum of solid angles subtended by the elementary triangles}$

$$= \omega_{\Delta AOB} + \omega_{\Delta BOC} + \omega_{\Delta COD} + \omega_{\Delta DOE} + \omega_{\Delta EOF} + \omega_{\Delta AOF}$$

While, the values of solid angles subtended by elementary triangles are determined by using **axiom of triangle** i.e. dividing each elementary triangle into two right triangles & using standard formula-1 of right triangle.

V. Axiom of Triangle

“If the perpendicular, drawn from a given point in the space to the plane of a given triangle, passes through one of the vertices then that triangle can be divided internally or externally (w.r.t. F.O.P.) into two right triangles having common vertex at the foot of perpendicular, simply by drawing a normal from the common vertex (i.e. F.O.P.) to the opposite side of given triangle.”

- An **acute angled triangle** is **internally divided** w.r.t. F.O.P. (i.e. common vertex)
- A **right angled triangle** is **internally divided** w.r.t. F.O.P. (i.e. common vertex)
- An **obtuse angled triangle** is **divided**

Internally if and only if the **angle of common vertex (F.O.P.) is obtuse**

Externally if and only if the **angle of common vertex (F.O.P.) is acute**

Now, consider a given point $P(0, 0, h)$ (located **perpendicular to the plane of paper**) lying at a height h on the normal axes passing through the vertices A, B & C where the points $P_1(0, 0, h), P_2(0, 0, h)$ & $P_3(0, 0, h)$ are the different locations of given point P on the perpendiculars passing through the vertices A, B & C respectively at the same height h .

(See the different cases in the figures (5), (6) & (7) below)

- **Acute Angled Triangle:**

Consider the given point $P_1(0, 0, h)$ at a normal height h from the vertex ‘A’ (i.e. foot of perpendicular drawn from the point P_1 to the plane of ΔABC)

Now, draw the perpendicular AM from F.O.P. ‘A’ to the opposite side BC to divide the ΔABC into two sub-elementary right triangles ΔAMB & ΔAMC

In this case, the area ($A_{\Delta ABC}$) of ΔABC is given by

$$\begin{aligned} A_{\Delta ABC} &= \text{algebraic sum of areas of elementary triangles} \\ &= A_{\Delta AMB} + A_{\Delta AMC} \quad (\Delta ABC \text{ is internally divided}) \end{aligned}$$

Hence replacing the areas by the corresponding values of solid angles, we get

$$\begin{aligned} \omega_{\Delta ABC} &= \text{algebraic sum of solid angles subtended by elementary triangles at point } P_1(0, 0, h) \\ &= \omega_{\Delta AMB} + \omega_{\Delta AMC} \quad (\Delta ABC \text{ is internally divided}) \end{aligned}$$

The values of $\omega_{\Delta AMB}$ & $\omega_{\Delta AMC}$ subtended by the right triangles ΔAMB & ΔAMC respectively are calculated by using standard formula-1 of right triangle (from eq(1) or eq(2)) as follows

$$\omega = \sin^{-1} \left\{ \frac{b}{\sqrt{b^2 + p^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{b^2 + p^2}} \right) \left(\frac{h}{\sqrt{h^2 + p^2}} \right) \right\}$$

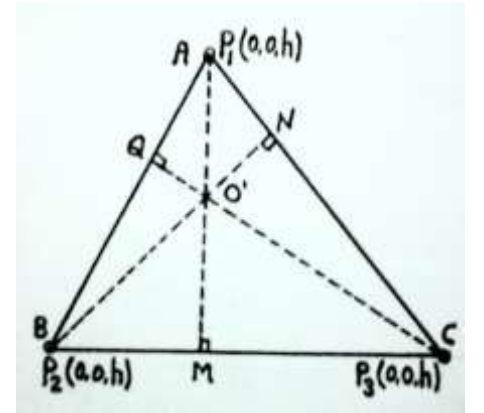


Fig 5: Acute Angled Triangle

Measure the dimensions AM, BM & P_1A from the drawing & set $b = BM, p = AM$ &

$h = \text{normal height of point } P_1(0, 0, h) \text{ from the vertex 'A' (i.e. F.O.P.)} = P_1A$ in above expression

We get, the solid angle subtended by the right ΔAMB at the given point $P_1(0, 0, h)$

$$\omega_{\Delta AMB} = \sin^{-1} \left\{ \frac{BM}{\sqrt{(BM)^2 + (AM)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{BM}{\sqrt{(BM)^2 + (AM)^2}} \right) \left(\frac{P_1A}{\sqrt{(P_1A)^2 + (AM)^2}} \right) \right\}$$

Similarly, solid angle subtended by the right ΔAMC at the the same point $P_1(0, 0, h)$

$$\omega_{\Delta AMC} = \sin^{-1} \left\{ \frac{CM}{\sqrt{(CM)^2 + (AM)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CM}{\sqrt{(CM)^2 + (AM)^2}} \right) \left(\frac{P_1A}{\sqrt{(P_1A)^2 + (AM)^2}} \right) \right\}$$

Hence, the solid angle subtended by the given ΔABC at the the given point $P_1(0, 0, h)$ is calculated as

$$\Rightarrow \omega_{\Delta ABC} = \omega_{\Delta AMB} + \omega_{\Delta AMC}$$

Similarly, for the location of given point $P_2(0, 0, h)$ lying at a normal height h from the vertex 'B' (i.e. foot of perpendicular drawn from the point $P_2(0, 0, h)$ to the plane of ΔABC) (See figure 5 above)

By following the above procedure, we get the solid angle subtended by the given ΔABC at the the given point $P_2(0, 0, h)$ as follows

$$\Rightarrow \omega_{\Delta ABC} = \omega_{\Delta ANB} + \omega_{\Delta CNB}$$

$$\omega_{\Delta ANB} = \sin^{-1} \left\{ \frac{AN}{\sqrt{(AN)^2 + (BN)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{AN}{\sqrt{(AN)^2 + (BN)^2}} \right) \left(\frac{P_2B}{\sqrt{(P_2B)^2 + (BN)^2}} \right) \right\} \&$$

$$\omega_{\Delta CNB} = \sin^{-1} \left\{ \frac{CN}{\sqrt{(CN)^2 + (BN)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CN}{\sqrt{(CN)^2 + (BN)^2}} \right) \left(\frac{P_2B}{\sqrt{(P_2B)^2 + (BN)^2}} \right) \right\}$$

Similarly, for the location of given point $P_3(0, 0, h)$ lying at a normal height h from the vertex 'C' (i.e. foot of perpendicular drawn from the point $P_3(0, 0, h)$ to the plane of ΔABC) (See figure 5 above)

By following the above procedure, we get the solid angle subtended by the given ΔABC at the the given point $P_3(0, 0, h)$ as follows

$$\Rightarrow \omega_{\Delta ABC} = \omega_{\Delta AQC} + \omega_{\Delta BQC}$$

We can calculate the corresponding values of $\omega_{\Delta AQC}$ & $\omega_{\Delta BQC}$ using standard formula-1 as follows

$$\omega_{\Delta AQC} = \sin^{-1} \left\{ \frac{AQ}{\sqrt{(AQ)^2 + (CQ)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{AQ}{\sqrt{(AQ)^2 + (CQ)^2}} \right) \left(\frac{P_3C}{\sqrt{(P_3C)^2 + (CQ)^2}} \right) \right\} \&$$

$$\omega_{\Delta BQC} = \sin^{-1} \left\{ \frac{BQ}{\sqrt{(BQ)^2 + (CQ)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{BQ}{\sqrt{(BQ)^2 + (CQ)^2}} \right) \left(\frac{P_3C}{\sqrt{(P_3C)^2 + (CQ)^2}} \right) \right\}$$

- **Right Angled Triangle:**

Consider the given point $P_1(0, 0, h)$ at a normal height h from the vertex 'A' (i.e. foot of perpendicular drawn from the point $P_1(0, 0, h)$ to the plane of right ΔBAC) (See the figure 6)

Now, draw the perpendicular AM from F.O.P. 'A' to the opposite side BC to divide the right ΔBAC into two sub-elementary right triangles ΔAMB & ΔAMC

In this case, the area ($A_{\Delta BAC}$) of right ΔBAC is given by
 $A_{\Delta BAC} = \text{algebraic sum of areas of elementary triangles}$

$$A_{\Delta BAC} = A_{\Delta AMB} + A_{\Delta AMC} \quad (\text{right } \Delta BAC \text{ is internally divided})$$

Hence replacing the areas by the corresponding values of solid angles, we get

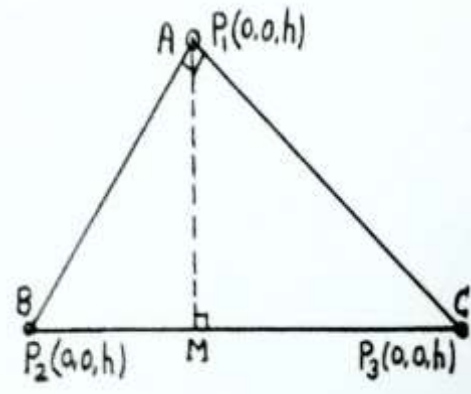


Fig 6: Right Angled Triangle

$$\begin{aligned} \omega_{\Delta BAC} &= \text{algebraic sum of solid angles subtended by elementary triangles at point } P_1(0,0,h) \\ &= \omega_{\Delta AMB} + \omega_{\Delta AMC} \quad (\text{right } \Delta BAC \text{ is internally divided}) \end{aligned}$$

The values of $\omega_{\Delta AMB}$ & $\omega_{\Delta AMC}$ subtended by the right triangles ΔAMB & ΔAMC respectively are calculated by using standard formula-1 of right triangle (from eq(1) or eq(2)) as follows

$$\omega = \sin^{-1} \left\{ \frac{b}{\sqrt{b^2 + p^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{b^2 + p^2}} \right) \left(\frac{h}{\sqrt{h^2 + p^2}} \right) \right\}$$

Measure the dimensions AM , BM & P_1A from the drawing & set $b = BM$, $p = AM$ &

$h = \text{normal height of point } P_1(0,0,h) \text{ from the vertex 'A' (i.e. F.O.P.)} = P_1A$ in above expression

We get, the solid angle subtended by the right ΔAMB at the given point $P_1(0,0,h)$

$$\omega_{\Delta AMB} = \sin^{-1} \left\{ \frac{BM}{\sqrt{(BM)^2 + (AM)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{BM}{\sqrt{(BM)^2 + (AM)^2}} \right) \left(\frac{P_1A}{\sqrt{(P_1A)^2 + (AM)^2}} \right) \right\}$$

Similarly, solid angle subtended by the right ΔAMC at the the same point $P_1(0,0,h)$

$$\omega_{\Delta AMC} = \sin^{-1} \left\{ \frac{CM}{\sqrt{(CM)^2 + (AM)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CM}{\sqrt{(CM)^2 + (AM)^2}} \right) \left(\frac{P_1A}{\sqrt{(P_1A)^2 + (AM)^2}} \right) \right\}$$

Hence, the solid angle subtended by right ΔBAC at the the given point $P_1(0,0,h)$ is calculated as

$$\Rightarrow \omega_{\Delta BAC} = \omega_{\Delta AMB} + \omega_{\Delta AMC}$$

Now, for the location of given point $P_2(0,0,h)$ lying at a normal height h from the vertex 'B' (i.e. foot of perpendicular drawn from the point $P_2(0,0,h)$ to the plane of right ΔBAC) (See figure 6 above)

Using standard formula-1 & setting $b = AC$, $p = AB$ &

$h = \text{normal height of point } P_2(0,0,h) \text{ from the vertex 'B' (i.e. F.O.P.)} = P_2B$

We get the solid angle subtended by the right ΔBAC at the the given point $P_2(0,0,h)$ as follows

$$\omega_{\Delta BAC} = \sin^{-1} \left\{ \frac{AC}{\sqrt{(AC)^2 + (AB)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{AC}{\sqrt{(AC)^2 + (AB)^2}} \right) \left(\frac{P_2B}{\sqrt{(P_2B)^2 + (AB)^2}} \right) \right\}$$

Similarly, for the location of given point $P_3(0, 0, h)$ lying at a normal height h from the vertex 'C' (i.e. foot of perpendicular drawn from the point $P_3(0, 0, h)$ to the plane of right $\triangle BAC$) (See figure 6 above)

Using standard formula-1 & setting $b = AB, p = AC$ &

$h = \text{normal height of point } P_3(0, 0, h) \text{ from the vertex 'C' (i.e. F.O.P.)} = P_3C$

We get the solid angle subtended by the right $\triangle BAC$ at the the given point $P_3(0, 0, h)$ as follows

$$\omega_{\triangle BAC} = \sin^{-1} \left\{ \frac{AB}{\sqrt{(AB)^2 + (AC)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{AB}{\sqrt{(AB)^2 + (AC)^2}} \right) \left(\frac{P_3C}{\sqrt{(P_3C)^2 + (AC)^2}} \right) \right\}$$

- **Obtuse Angled Triangle:**

Consider the given point $P_1(0, 0, h)$ at a normal height h from the vertex 'A' (i.e. foot of perpendicular drawn from the point $P_1(0, 0, h)$ to the plane of $\triangle ABC$) (See the figure 7)

Now, draw the perpendicular AM from F.O.P. 'A' to the opposite side BC to divide the obtuse $\triangle ABC$ into two sub-elementary right triangles $\triangle AMB$ & $\triangle AMC$

In this case, the area ($A_{\triangle ABC}$) of obtuse $\triangle ABC$ is given by

$A_{\triangle ABC} =$
algebraic sum of areas of elementary triangles

$$= A_{\triangle AMB} + A_{\triangle AMC} \quad (\text{obtuse } \triangle ABC \text{ is internally divided})$$

Hence replacing the areas by the corresponding values of solid angles, we get

$\omega_{\triangle ABC} = \text{algebraic sum of solid angles subtended by elementary triangles at point } P_1(0, 0, h)$

$$= \omega_{\triangle AMB} + \omega_{\triangle AMC} \quad (\text{obtuse } \triangle ABC \text{ is internally divided})$$

The values of $\omega_{\triangle AMB}$ & $\omega_{\triangle AMC}$ subtended by the right triangles $\triangle AMB$ & $\triangle AMC$ respectively are calculated by using standard formula-1 of right triangle (from eq(1) or eq(2)) as follows

$$\omega = \sin^{-1} \left\{ \frac{b}{\sqrt{b^2 + p^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{b^2 + p^2}} \right) \left(\frac{h}{\sqrt{h^2 + p^2}} \right) \right\}$$

Measure the dimensions AM, BM & P_1A from the drawing & set $b = BM, p = AM$ &

$h = \text{normal height of point } P_1(0, 0, h) \text{ from the vertex 'A' (i.e. F.O.P.)} = P_1A$ in above expression

We get, the solid angle subtended by the right $\triangle AMB$ at the given point $P_1(0, 0, h)$

$$\omega_{\triangle AMB} = \sin^{-1} \left\{ \frac{BM}{\sqrt{(BM)^2 + (AM)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{BM}{\sqrt{(BM)^2 + (AM)^2}} \right) \left(\frac{P_1A}{\sqrt{(P_1A)^2 + (AM)^2}} \right) \right\}$$

Similarly, solid angle subtended by the right $\triangle AMC$ at the the same point $P_1(0, 0, h)$

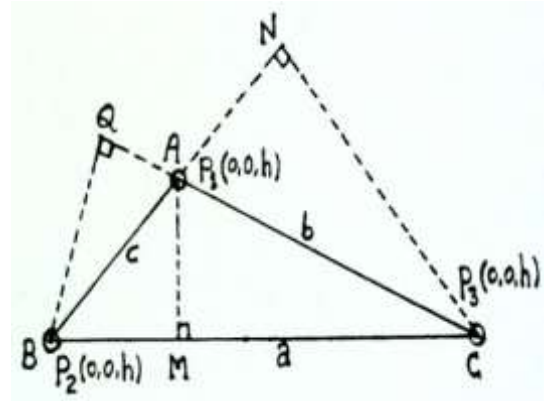


Fig 7: Obtuse Angled Triangle

$$\omega_{\Delta AMC} = \sin^{-1} \left\{ \frac{CM}{\sqrt{(CM)^2 + (AM)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CM}{\sqrt{(CM)^2 + (AM)^2}} \right) \left(\frac{P_1 A}{\sqrt{(P_1 A)^2 + (AM)^2}} \right) \right\}$$

Hence, the solid angle subtended by obtuse ΔABC at the the given point $P_1(0, 0, h)$ is calculated as

$$\Rightarrow \omega_{\Delta ABC} = \omega_{\Delta AMB} + \omega_{\Delta AMC}$$

Now, for the location of given point $P_2(0, 0, h)$ lying at a normal height h from the vertex 'B' (i.e. foot of perpendicular drawn from the point $P_2(0, 0, h)$ to the plane of ΔABC) (See figure 7 above)

Draw the perpendicular BQ from F.O.P. 'B' to the opposite side AC to divide the obtuse ΔABC into two sub-elementary right triangles ΔBQC & ΔBQA

In this case, the area ($A_{\Delta ABC}$) of obtuse ΔABC is given by

$$\begin{aligned} A_{\Delta ABC} &= \text{algebraic sum of areas of elementary triangles} \\ &= A_{\Delta BQC} - A_{\Delta BQA} \quad (\text{obtuse } \Delta ABC \text{ is externally divided}) \end{aligned}$$

Hence replacing the areas by the corresponding values of solid angles, we get

$$\begin{aligned} \omega_{\Delta ABC} &= \text{algebraic sum of solid angles subtended by elementary triangles at point } P_2(0, 0, h) \\ &= \omega_{\Delta BQC} - \omega_{\Delta BQA} \quad (\text{obtuse } \Delta ABC \text{ is externally divided}) \end{aligned}$$

The values of $\omega_{\Delta BQC}$ & $\omega_{\Delta BQA}$ subtended by the right triangles ΔBQC & ΔBQA respectively are calculated by using standard formula-1 of right triangle (from eq(1) or eq(2)) as follows

$$\omega = \sin^{-1} \left\{ \frac{b}{\sqrt{b^2 + p^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{b^2 + p^2}} \right) \left(\frac{h}{\sqrt{h^2 + p^2}} \right) \right\}$$

Measure the dimensions QC, BQ & P_2B from the drawing & set $b = QC, p = BQ$ &

$h = \text{normal height of point } P_2(0, 0, h) \text{ from the vertex 'B' (i.e. F.O.P.)} = P_2B$ in above expression

We get, the solid angle subtended by the right ΔBQC at the given point $P_2(0, 0, h)$

$$\omega_{\Delta BQC} = \sin^{-1} \left\{ \frac{QC}{\sqrt{(QC)^2 + (BQ)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{QC}{\sqrt{(QC)^2 + (BQ)^2}} \right) \left(\frac{P_2B}{\sqrt{(P_2B)^2 + (BQ)^2}} \right) \right\}$$

Similarly, solid angle subtended by the right ΔBQA at the the same point $P_2(0, 0, h)$

$$\omega_{\Delta BQA} = \sin^{-1} \left\{ \frac{QA}{\sqrt{(QA)^2 + (BQ)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{QA}{\sqrt{(QA)^2 + (BQ)^2}} \right) \left(\frac{P_2B}{\sqrt{(P_2B)^2 + (BQ)^2}} \right) \right\}$$

Hence, the solid angle subtended by obtuse ΔABC at the the given point $P_2(0, 0, h)$ is calculated as

$$\Rightarrow \omega_{\Delta ABC} = \omega_{\Delta BQC} - \omega_{\Delta BQA}$$

Similarly, for the location of given point $P_3(0, 0, h)$ lying at a normal height h from the vertex 'C' (i.e. foot of perpendicular drawn from the point $P_3(0, 0, h)$ to the plane of ΔABC) (See figure 7 above)

Following the above procedure, solid angle subtended by obtuse ΔABC at the the given point $P_3(0, 0, h)$ is calculated as

$$\Rightarrow \omega_{\Delta ABC} = \omega_{\Delta CNB} - \omega_{\Delta CNA} \quad (\text{obtuse } \Delta ABC \text{ is externally divided})$$

$$\text{where, } \omega_{\Delta CNB} = \sin^{-1} \left\{ \frac{BN}{\sqrt{(BN)^2 + (CN)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{BN}{\sqrt{(BN)^2 + (CN)^2}} \right) \left(\frac{P_3C}{\sqrt{(P_3C)^2 + (CN)^2}} \right) \right\} \&$$

$$\Rightarrow \omega_{\Delta CNA} = \sin^{-1} \left\{ \frac{AN}{\sqrt{(AN)^2 + (CN)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{AN}{\sqrt{(AN)^2 + (CN)^2}} \right) \left(\frac{P_3C}{\sqrt{(P_3C)^2 + (CN)^2}} \right) \right\}$$

Now, we will apply the same procedure in case of any polygonal plane by dividing it into the elementary triangles with common vertex at the F.O.P. & then each elementary triangle into two sub-elementary right triangles with common vertex at the F.O.P. Now, it's easy to understand the theory of polygon

VI. Axiom of polygon

“For a given point in the space, each of the polygons can be divided internally or externally or both w.r.t. foot of perpendicular (F.O.P.) drawn from the given point to the plane of polygon into a certain number of the elementary triangles, having a common vertex at the foot of perpendicular, by joining all the vertices of polygon to the F.O.P. by straight lines (generally extended).”

- Polygon is **externally divided** if the **F.O.P.** lies **outside the boundary**
- Polygon is divided **internally or externally or both** if the **F.O.P.** lies **inside or on the boundary** depending on the **geometrical shape of polygon (i.e. angles & sides)**

(See the different cases in the figures (8), (9), (10) & (11) below as explained in Theory of Polygon)

Elementary Triangle: Any of the elementary triangles obtained by joining all the vertices of polygon to the foot of perpendicular drawn from the given point to the plane of polygon such that all the elementary triangles have common vertex at the foot of perpendicular (**F.O.P.**) is called elementary triangle.

Sub-elementary Right Triangles: These are the right triangles which are obtained by drawing a perpendicular from F.O.P. to the opposite side in any of the elementary triangles. Thus one elementary triangle is internally or externally divided into two sub-elementary right triangles. These sub-elementary right triangles always have common vertex at the foot of perpendicular (F.O.P.)

VII. HCR's Theory of Polygon (Proposed by the Author-2014)

This theory is applicable for any polygonal plane (i.e. **plane bounded by the straight lines only**) & any point in the space if the following parameters are already known

1. **Geometrical shape & dimensions of the polygonal plane**
2. **Normal distance (h) of the given point from the plane of polygon**

3. Location of foot of perpendicular (F.O.P.) drawn from given point to the plane of polygon

According to this theory “Solid angle subtended by any polygonal plane at any point in the space is the algebraic sum of solid angles subtended at the same point by all the elementary triangles (obtained by joining all the vertices of polygon to the foot of perpendicular) having common vertex at the foot of perpendicular drawn from the given point to the plane of polygon such that algebraic sum of areas of all these triangles is equal to the area of given polygon.” It has no mathematical proof.

Mathematically, solid angle ($\omega_{polygon}$) subtended by any polygonal plane with ‘n’ number of sides & area ‘A’ at any point in the space is given by

$$\omega_{polygon} = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 \dots \dots \dots$$

$$\Rightarrow \omega_{polygon} = \left[\sum \omega_i \right]_{algebraic}$$

$$where, \quad A_{polygon} = A_1 + A_2 + A_3 + A_4 + A_5 \dots \dots \dots = \left[\sum A_i \right]_{algebraic}$$

$A_1, A_2, A_3, A_4, A_5, \dots$ are the areas of elementary triangles subtending the solid angles $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5 \dots$ at the given point which are determined by using standard formula-1 (as given from eq(1)) along with the necessary dimensions which are found out analytically or by tracing the diagram which is easier.

- **Element Method:** It the method of **diving a polygon internally or externally into sub-elementary right triangles** having **common vertex at the F.O.P.** drawn from the given point to the plane of polygon such that **algebraic sum of areas** of all these triangles is equal to the **area of given polygon.**”
- **Working Steps:**

STEP 1: Trace/draw the diagram of the given polygon (plane) with known geometrical dimensions.

STEP 2: Specify the location of foot of perpendicular drawn from a given point to the plane of polygon.

STEP 3: Join all the vertices of polygon to the foot of perpendicular by the extended straight lines. Thus the polygon is divided into a number of elementary triangles having a common vertex at the foot of perpendicular.

STEP 4: Further, consider each elementary triangle & divide it internally or externally into two sub-elementary right triangles simply by drawing a perpendicular from F.O.P. (i.e. common vertex) to the opposite side of that elementary triangle.

STEP 5: Now, find out the area of given polygon as the algebraic sum of areas of all these sub-elementary right triangles i.e. area of each of right triangles must be taken with proper sign whose sum gives area of polygon.

Remember: All the elementary triangles & sub-elementary right triangles must have their one vertex common at the foot of perpendicular (F.O.P.).

STEP 6: Replace areas of all these sub-elementary right triangles by their respective values of solid angle subtended at the given point in the space.

STEP 7: Calculate solid angle subtended by each of individual sub-elementary right triangles by using the standard formula-1 of right triangle (from eq(1) as derived above) as follows

$$\omega = \sin^{-1} \left\{ \frac{b}{\sqrt{b^2 + p^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{b^2 + p^2}} \right) \left(\frac{h}{\sqrt{h^2 + p^2}} \right) \right\}$$

STEP 8: Now, solid angle subtended by polygonal plane at the given point will be the algebraic sum of all its individual sub-elementary right triangles as given (By **Element Method**)

$$\omega_{polygon} = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \dots \dots \dots$$

* **Finally**, we get the value of $\omega_{polygon}$ as the **algebraic sum** of solid angles subtended by **right triangles only**.

VIII. Special Cases for a Polygonal Plane

Let's us consider a polygonal plane (with vertices) 123456 & a given point say $P(0, 0, h)$ at a normal height h from the given plane in the space.

Now, specify the location of foot of perpendicular say 'O' on the plane of polygon which may lie

1. **Outside the boundary**
2. **Inside the boundary**
3. **On the boundary**
 - a. **On one of the sides** or b. **At one of the vertices**

Let's consider the above cases one by one as follows

1. F.O.P. outside the boundary:

Let the foot of perpendicular 'O' lie outside the boundary of polygon. Join all the vertices of polygon (plane) 123456 to the foot of perpendicular 'O' by the extended straight lines

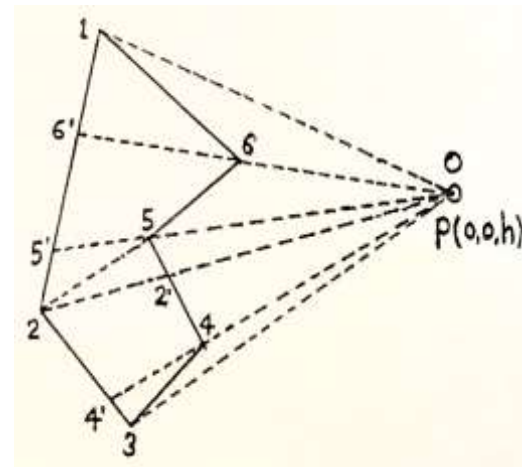


Fig 8: F.O.P. lying outside the boundary

(As shown in the figure 8) Thus the polygon is divided into elementary triangles (obtained by the extension lines), all having common vertex at the foot of perpendicular 'O'.

Now the solid angle subtended by the polygonal plane at the given point 'P' in the space is given

By Element-Method

$$\omega_{123456} = \omega_{616'} + \omega_{566'5'} + \omega_{22'55'} + \omega_{4'42'2} + \omega_{344'} \dots \dots \dots (I)$$

$$\text{where, } \omega_{616'} = \omega_{6'O1} - \omega_{6'O1}$$

$$\omega_{566'5'} = \omega_{5'O6'} - \omega_{5'O6} \quad \omega_{22'55'} = \omega_{2'O5'} - \omega_{2'O5}$$

$$\omega_{4'42'2} = \omega_{4'O2} - \omega_{4'O2'}$$

$$\omega_{344'} = \omega_{3'O4'} - \omega_{3'O4}$$

Thus, the value of solid angle subtended by the polygon at the given point is obtained by setting these values in the eq. (I) as follows

$$\Rightarrow \omega_{123456} = (\omega_{6'O1} - \omega_{6'O1}) + (\omega_{5'O6'} - \omega_{5'O6}) + (\omega_{2'O5'} - \omega_{2'O5}) + (\omega_{4'O2} - \omega_{4'O2'})$$

Further, each of the individual elementary triangles is divided into two sub-elementary right triangles for which the values of solid angle are determined by using standard formula-1 of right triangle.

2. F.O.P. inside the boundary:

Let the foot of perpendicular 'O' lie inside the boundary of polygon. Join all the vertices of polygon (plane) 123456 to the foot of perpendicular 'O' by the extended straight lines (As shown in the figure 9)

Thus the polygon is divided into elementary triangles (obtained by the extension lines), all having common vertex at the foot of perpendicular 'O'. Now the solid angle subtended by the polygonal plane at the given point P in the space is given

By Element-Method

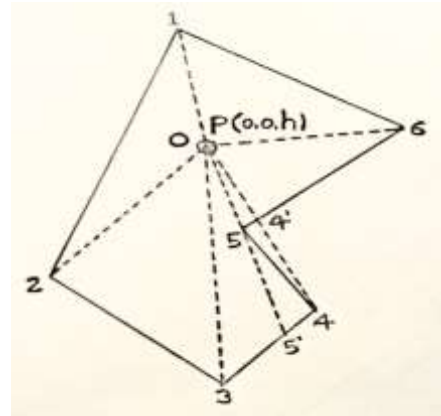


Figure 9: F.O.P. lying inside the boundary

$$\omega_{123456} = \omega_{102} + \omega_{203} + \omega_{305'} + \omega_{5'54} + \omega_{504'} \dots \dots \dots (II)$$

Where,

$$\omega_{5'54} = \omega_{5'04} - \omega_{504}$$

Thus, the value of solid angle subtended by the polygon at the given point is obtained by setting these values in the eq. (II) as follows

$$\Rightarrow \omega_{123456} = \omega_{102} + \omega_{203} + \omega_{305'} + (\omega_{5'04} - \omega_{504}) + \omega_{504'}$$

Further, each of the individual elementary triangles is divided into two sub-elementary right triangles for which the values of solid angle are determined by using standard formula-1 of right triangle.

3. F.O.P. on the boundary: Further two cases are possible

a. F.O.P. lying on one of the sides:

Let the foot of perpendicular 'O' lie on one of the sides say '12' of polygon. Join all the vertices of polygon (plane) 123456 to the foot of perpendicular 'O' by the straight lines

(As shown in the figure 10)

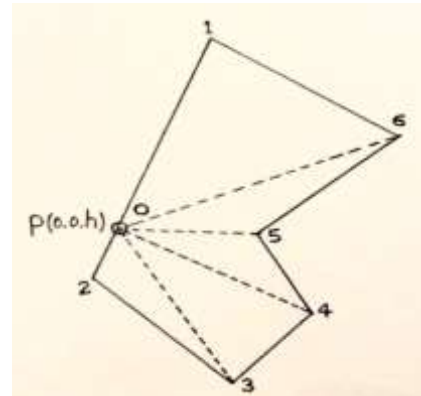


Figure 10: F.O.P. lying at one of the sides

Thus the polygon is divided into elementary triangles (obtained by the straight lines), all having common vertex at the foot of perpendicular 'O'. Now the solid angle subtended by the polygonal plane at the given point 'P' in the space is given By Element-Method

$$\omega_{123456} = \omega_{106} + \omega_{605} + \omega_{504} + \omega_{403} + \omega_{302} \dots \dots \dots (III)$$

Further, each of the individual elementary triangles is divided into two sub-elementary right triangles for which the values of solid angle are determined by using standard formula-1 of right triangle.

b. F.O.P. lying at one of the vertices:

Let the foot of perpendicular lie on one of the vertices say '5' of polygon. Join all the vertices of polygon (plane) 123456 to the foot of perpendicular (i.e. common vertex '5') by the straight lines

(As shown in the figure 11)

Thus the polygon is divided into elementary triangles (obtained by the straight lines), all having common vertex at the foot of perpendicular '5'. Now the solid angle subtended by the polygonal plane at the given point 'P' in the space is given

By Element-Method

$$\omega_{123456} = \omega_{152} + \omega_{253} + \omega_{354} + \omega_{456} \dots \dots \dots (IV)$$

Further, each of the individual elementary triangles is divided into two sub-elementary right triangles for which the values of solid angle are determined by using standard formula-1 of right triangle.

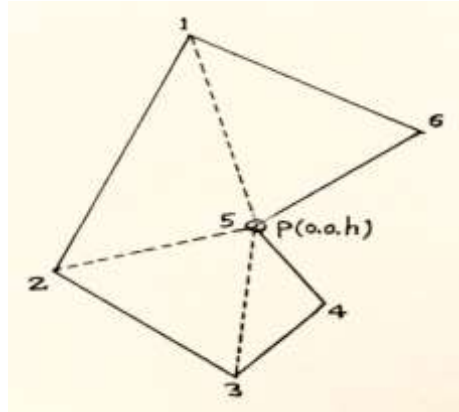


Fig 11: F.O.P. lying on one of the vertices

IX. Analytical Applications of Theory of Polygon

Let's take some examples of polygonal plane to find out the solid angle at different locations of a given point in the space. For ease of understanding, we will take some particular examples where division of polygons into right triangles is easier & standard formula can directly be applied by analytical measurements of necessary dimensions i.e. drawing is not required. Although, random location of point may cause complex calculations

• Right Triangular Plane

F.O.P. lying on the right angled vertex: Let there be a right triangular plane ABC having orthogonal sides AB = a & BC = b & a given point say P(0, 0, h) at a normal height h from the right angled vertex 'B'

(As shown in the figure 12 below)

Now, draw a perpendicular BN from the vertex 'B' to the hypotenuse AC to divide the right $\triangle ABC$ into elementary right triangles $\triangle ANB$ & $\triangle BNC$. By element method, solid angle subtended by the right $\triangle ABC$ at the given point P

$$\Rightarrow \omega_{\triangle ABC} = \omega_{\triangle ANB} + \omega_{\triangle BNC}$$

Where, the values of $\omega_{\triangle ANB}$ & $\omega_{\triangle BNC}$ are calculated by standard formula-1 as follows

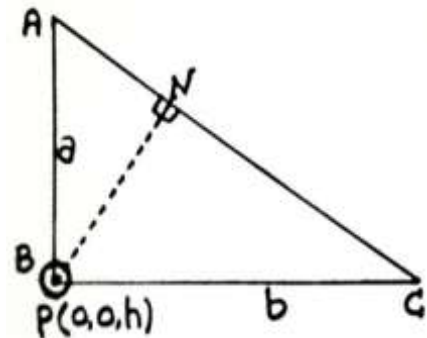


Fig 12: Point P lying on the normal axis passing through the right angled vertex B

Values of BN, AN & CN can be easily calculated as follows

$$BN = \frac{ab}{\sqrt{a^2 + b^2}}, \quad AN = \frac{a^2}{\sqrt{a^2 + b^2}} \quad \& \quad CN = \frac{b^2}{\sqrt{a^2 + b^2}}$$

$$\therefore \omega_{\triangle ANB} = \sin^{-1} \left\{ \frac{AN}{\sqrt{(AN)^2 + (BN)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{AN}{\sqrt{(AN)^2 + (BN)^2}} \right) \left(\frac{PB}{\sqrt{(PB)^2 + (BN)^2}} \right) \right\}$$

$$\begin{aligned}
&= \sin^{-1} \left\{ \frac{\frac{a^2}{\sqrt{a^2+b^2}}}{\sqrt{\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{ab}{\sqrt{a^2+b^2}}\right)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{\frac{a^2}{\sqrt{a^2+b^2}}}{\sqrt{\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{ab}{\sqrt{a^2+b^2}}\right)^2}} \right) \left(\frac{h}{\sqrt{(h)^2 + \left(\frac{ab}{\sqrt{a^2+b^2}}\right)^2}} \right) \right\} \\
&\Rightarrow \omega_{\Delta ANB} = \sin^{-1} \left\{ \frac{a}{\sqrt{a^2+b^2}} \right\} - \sin^{-1} \left\{ \frac{ah}{\sqrt{h^2(a^2+b^2) + a^2b^2}} \right\}
\end{aligned}$$

Similarly, we can calculate solid angle subtended by the ΔCNB at the given point $P(0, 0, h)$

$$\begin{aligned}
\omega_{\Delta CNB} &= \sin^{-1} \left\{ \frac{CN}{\sqrt{(CN)^2 + (BN)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CN}{\sqrt{(CN)^2 + (BN)^2}} \right) \left(\frac{PB}{\sqrt{(PB)^2 + (BN)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{\frac{b^2}{\sqrt{a^2+b^2}}}{\sqrt{\left(\frac{b^2}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{ab}{\sqrt{a^2+b^2}}\right)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{\frac{b^2}{\sqrt{a^2+b^2}}}{\sqrt{\left(\frac{b^2}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{ab}{\sqrt{a^2+b^2}}\right)^2}} \right) \left(\frac{h}{\sqrt{(h)^2 + \left(\frac{ab}{\sqrt{a^2+b^2}}\right)^2}} \right) \right\} \\
&\Rightarrow \omega_{\Delta CNB} = \sin^{-1} \left\{ \frac{b}{\sqrt{a^2+b^2}} \right\} - \sin^{-1} \left\{ \frac{bh}{\sqrt{h^2(a^2+b^2) + a^2b^2}} \right\}
\end{aligned}$$

Hence, the solid angle subtended by given right ΔABC at the given point $P(0, 0, h)$ lying at a normal height h from the right angled vertex 'B' is calculated as follows

$$\begin{aligned}
&\Rightarrow \omega_{\Delta ABC} = \omega_{\Delta ANB} + \omega_{\Delta CNB} \\
&= \sin^{-1} \left\{ \frac{a}{\sqrt{a^2+b^2}} \right\} - \sin^{-1} \left\{ \frac{ah}{\sqrt{h^2(a^2+b^2) + a^2b^2}} \right\} + \sin^{-1} \left\{ \frac{b}{\sqrt{a^2+b^2}} \right\} - \sin^{-1} \left\{ \frac{bh}{\sqrt{h^2(a^2+b^2) + a^2b^2}} \right\} \\
&= \left[\sin^{-1} \left\{ \frac{a}{\sqrt{a^2+b^2}} \right\} + \sin^{-1} \left\{ \frac{b}{\sqrt{a^2+b^2}} \right\} \right] - \left[\sin^{-1} \left\{ \frac{ah}{\sqrt{h^2(a^2+b^2) + a^2b^2}} \right\} + \sin^{-1} \left\{ \frac{bh}{\sqrt{h^2(a^2+b^2) + a^2b^2}} \right\} \right]
\end{aligned}$$

Using, $\sin^{-1} x + \sin^{-1} y = \sin^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2}) \quad \forall (-1 \leq (x, y) \leq 1)$ & simplifying, we get

$$\begin{aligned}
&\Rightarrow \omega_{\Delta ABC} = \sin^{-1} \left\{ \frac{a^2+b^2}{a^2+b^2} \right\} - \sin^{-1} \left\{ \frac{h(a^2\sqrt{h^2+b^2} + b^2\sqrt{h^2+a^2})}{h^2(a^2+b^2) + a^2b^2} \right\} \\
&= \frac{\pi}{2} - \sin^{-1} \left\{ \frac{h(a^2\sqrt{h^2+b^2} + b^2\sqrt{h^2+a^2})}{h^2(a^2+b^2) + a^2b^2} \right\} \\
&\therefore \omega_{\Delta ABC} = \cos^{-1} \left\{ \frac{h(a^2\sqrt{h^2+b^2} + b^2\sqrt{h^2+a^2})}{h^2(a^2+b^2) + a^2b^2} \right\} \quad \dots \dots \dots (3)
\end{aligned}$$

Note: This is the standard formula to find out the value of solid angle subtended by a right triangular plane, with orthogonal sides a & b , at any point lying at a normal height h from the right angled vertex.

- **Rectangular Plane**

F.O.P. lying on one of the vertices of rectangular plane: Let there be a rectangular plane ABCD having length $AB = l$ & width $BC = b$ & a given point say $P(0, 0, h)$ at a normal height h from any of the vertices say vertex 'A'

Now, draw a perpendicular PA from the given point 'P' to the plane of rectangle ABCD passing through the vertex A (i.e. foot of perpendicular). Join the vertex 'C' to the F.O.P. 'A' to divide the plane into elementary triangles i.e. right triangles ΔABC & ΔADC .

It is clear from the figure (13) that the area of rectangle ABCD

$$A_{ABCD} = A_{\Delta ABC} + A_{\Delta ADC}$$

Hence, using **Element Method** by replacing areas by corresponding values of solid angles, the solid angle subtended by the rectangular plane ABCD at the given point P

$$\omega_{ABCD} = \omega_{\Delta ABC} + \omega_{\Delta ADC}$$

Now, using standard formula-1, solid angle subtended by the right ΔABC at the point $P(0, 0, h)$

$$\begin{aligned}\omega_{\Delta ABC} &= \sin^{-1} \left\{ \frac{BC}{\sqrt{(BC)^2 + (AB)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{BC}{\sqrt{(BC)^2 + (AB)^2}} \right) \left(\frac{PA}{\sqrt{(PA)^2 + (AB)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{b}{\sqrt{b^2 + l^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{b^2 + l^2}} \right) \left(\frac{h}{\sqrt{h^2 + l^2}} \right) \right\} \quad \& \\ \omega_{\Delta ADC} &= \sin^{-1} \left\{ \frac{CD}{\sqrt{(CD)^2 + (AD)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CD}{\sqrt{(CD)^2 + (AD)^2}} \right) \left(\frac{PA}{\sqrt{(PA)^2 + (AD)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{l}{\sqrt{l^2 + b^2}} \right\} - \sin^{-1} \left\{ \left(\frac{l}{\sqrt{l^2 + b^2}} \right) \left(\frac{h}{\sqrt{h^2 + b^2}} \right) \right\}\end{aligned}$$

Hence, the solid angle subtended by rectangular plane ABCD at the given point $P(0, 0, h)$ lying at a normal height h from vertex A

$$\begin{aligned}\omega_{ABCD} &= \omega_{\Delta ABC} + \omega_{\Delta ADC} \\ &= \sin^{-1} \left\{ \frac{b}{\sqrt{b^2 + l^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{b^2 + l^2}} \right) \left(\frac{h}{\sqrt{h^2 + l^2}} \right) \right\} + \sin^{-1} \left\{ \frac{l}{\sqrt{l^2 + b^2}} \right\} - \sin^{-1} \left\{ \left(\frac{l}{\sqrt{l^2 + b^2}} \right) \left(\frac{h}{\sqrt{h^2 + b^2}} \right) \right\} \\ &= \left[\sin^{-1} \left\{ \frac{l}{\sqrt{l^2 + b^2}} \right\} + \sin^{-1} \left\{ \frac{b}{\sqrt{b^2 + l^2}} \right\} \right] - \left[\sin^{-1} \left\{ \left(\frac{l}{\sqrt{l^2 + b^2}} \right) \left(\frac{h}{\sqrt{h^2 + b^2}} \right) \right\} + \sin^{-1} \left\{ \left(\frac{b}{\sqrt{b^2 + l^2}} \right) \left(\frac{h}{\sqrt{h^2 + l^2}} \right) \right\} \right]\end{aligned}$$

Using, $\sin^{-1} x + \sin^{-1} y = \sin^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2}) \quad \forall (-1 \leq (x, y) \leq 1)$ & simplifying, we get

$$\begin{aligned}\Rightarrow \omega_{ABCD} &= \sin^{-1} \left\{ \frac{l^2 + b^2}{l^2 + b^2} \right\} - \sin^{-1} \left\{ \frac{h(l^2 + b^2)\sqrt{l^2 + b^2 + h^2}}{(l^2 + b^2)\sqrt{l^2 + h^2}\sqrt{b^2 + h^2}} \right\} \\ &= \frac{\pi}{2} - \sin^{-1} \left\{ \frac{h\sqrt{l^2 + b^2 + h^2}}{\sqrt{l^2 + h^2}\sqrt{b^2 + h^2}} \right\} = \cos^{-1} \left\{ \frac{h\sqrt{l^2 + b^2 + h^2}}{\sqrt{l^2 + h^2}\sqrt{b^2 + h^2}} \right\} = \sin^{-1} \left\{ \frac{lb}{\sqrt{l^2 + h^2}\sqrt{b^2 + h^2}} \right\} \\ \therefore \omega_{ABCD} &= \sin^{-1} \left\{ \frac{lb}{\sqrt{(l^2 + h^2)(b^2 + h^2)}} \right\} \quad \dots \dots \dots (4)\end{aligned}$$

Note: This is the standard formula to find out the value of solid angle subtended by a rectangular plane of size $a \times b$ at any point lying at a normal height h from any of the vertices.

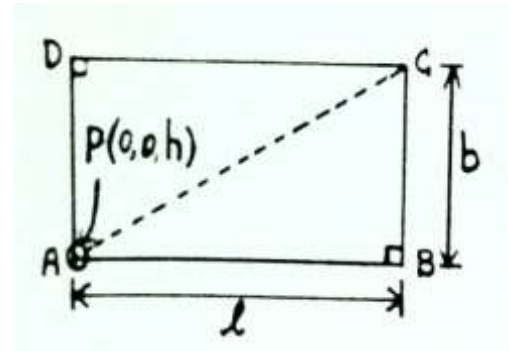


Fig 13: Point P lying on the normal axis passing through one of the vertices of rectangular plane

F.O.P. lying on the centre of rectangular plane: Let the given point $P(0, 0, h)$ be lying at a normal height h from the centre 'O' (i.e. F.O.P.) of rectangular plane ABCD. Join all the vertices A, B, C, D to the F.O.P. 'O'.

Thus, rectangle ABCD is divided into four elementary triangles $\Delta AOB, \Delta BOC, \Delta COD$ & ΔAOD . Hence, Area of rectangle ABCD

$$\begin{aligned}\Rightarrow A_{ABCD} &= A_{\Delta AOB} + A_{\Delta BOC} + A_{\Delta COD} + A_{\Delta AOD} \\ &= 2(A_{\Delta AOB} + A_{\Delta AOD}) \quad \text{By symmetry}\end{aligned}$$

Now, draw perpendiculars OE & OF from the F.O.P. 'O' to the opposite sides AB in ΔAOB & AD in ΔAOD to divide them into sub-elementary right triangles $\Delta OEA, \Delta OEB$ in ΔAOB &

$\Delta OFA, \Delta OFD$ in ΔAOD

$$\Rightarrow A_{ABCD} = 2 \times 2(A_{\Delta OEA} + A_{\Delta OFA}) = 4(A_{\Delta OEA} + A_{\Delta OFA}) \quad \text{By symmetry}$$

Hence, using **Element Method** by replacing areas by corresponding values of solid angles, the solid angle subtended by the rectangular plane ABCD at the given point P

$$\omega_{ABCD} = 4(\omega_{\Delta OEA} + \omega_{\Delta OFA})$$

Now, using standard formula-1, solid angle subtended by the right ΔOEA at the point $P(0, 0, h)$

$$\begin{aligned}\omega_{\Delta OEA} &= \sin^{-1} \left\{ \frac{AE}{\sqrt{(AE)^2 + (OE)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{AE}{\sqrt{(AE)^2 + (OE)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OE)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{\left(\frac{l}{2}\right)}{\sqrt{\left(\frac{l}{2}\right)^2 + \left(\frac{b}{2}\right)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{\left(\frac{l}{2}\right)}{\sqrt{\left(\frac{l}{2}\right)^2 + \left(\frac{b}{2}\right)^2}} \right) \left(\frac{h}{\sqrt{h^2 + \left(\frac{b}{2}\right)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{l}{\sqrt{l^2 + b^2}} \right\} - \sin^{-1} \left\{ \left(\frac{l}{\sqrt{l^2 + b^2}} \right) \left(\frac{2h}{\sqrt{4h^2 + b^2}} \right) \right\} \quad \& \\ \omega_{\Delta OFA} &= \sin^{-1} \left\{ \frac{AF}{\sqrt{(AF)^2 + (OF)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{AF}{\sqrt{(AF)^2 + (OF)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OF)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{\left(\frac{b}{2}\right)}{\sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{l}{2}\right)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{\left(\frac{b}{2}\right)}{\sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{l}{2}\right)^2}} \right) \left(\frac{h}{\sqrt{h^2 + \left(\frac{l}{2}\right)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{b}{\sqrt{l^2 + b^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{l^2 + b^2}} \right) \left(\frac{2h}{\sqrt{4h^2 + l^2}} \right) \right\}\end{aligned}$$

Hence, the solid angle subtended by rectangular plane ABCD at the given point $P(0, 0, h)$ lying at a normal height h from vertex A

$$\begin{aligned}\omega_{ABCD} &= 4(\omega_{\Delta OEA} + \omega_{\Delta OFA}) \\ &= 4 \left[\sin^{-1} \left\{ \frac{l}{\sqrt{l^2 + b^2}} \right\} - \sin^{-1} \left\{ \left(\frac{l}{\sqrt{l^2 + b^2}} \right) \left(\frac{2h}{\sqrt{4h^2 + b^2}} \right) \right\} + \sin^{-1} \left\{ \frac{b}{\sqrt{l^2 + b^2}} \right\} - \sin^{-1} \left\{ \left(\frac{b}{\sqrt{l^2 + b^2}} \right) \left(\frac{2h}{\sqrt{4h^2 + l^2}} \right) \right\} \right]\end{aligned}$$

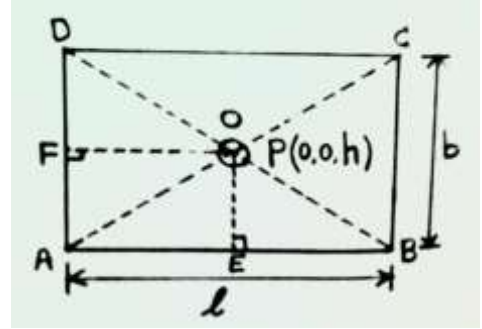


Fig 14: Point P lying on the normal axis passing through the centre of rectangular plane

$$= 4 \left[\sin^{-1} \left\{ \frac{l}{\sqrt{l^2 + b^2}} \right\} + \sin^{-1} \left\{ \frac{b}{\sqrt{l^2 + b^2}} \right\} \right] - 4 \left[\sin^{-1} \left\{ \left(\frac{l}{\sqrt{l^2 + b^2}} \right) \left(\frac{2h}{\sqrt{4h^2 + b^2}} \right) \right\} + \sin^{-1} \left\{ \left(\frac{b}{\sqrt{l^2 + b^2}} \right) \left(\frac{2h}{\sqrt{4h^2 + l^2}} \right) \right\} \right]$$

Using, $\sin^{-1} x + \sin^{-1} y = \sin^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2}) \quad \forall (-1 \leq (x, y) \leq 1)$ & simplifying, we get

$$\begin{aligned} \Rightarrow \omega_{ABCD} &= 4 \left[\sin^{-1} \left\{ \frac{l^2 + b^2}{l^2 + b^2} \right\} - \sin^{-1} \left\{ \frac{2h(l^2 + b^2)\sqrt{l^2 + b^2 + 4h^2}}{(l^2 + b^2)\sqrt{l^2 + 4h^2}\sqrt{b^2 + 4h^2}} \right\} \right] \\ &= 4 \left[\frac{\pi}{2} - \sin^{-1} \left\{ \frac{2h\sqrt{l^2 + b^2 + 4h^2}}{\sqrt{l^2 + 4h^2}\sqrt{b^2 + 4h^2}} \right\} \right] \\ &= 4 \cos^{-1} \left\{ \frac{2h\sqrt{l^2 + b^2 + 4h^2}}{\sqrt{l^2 + 4h^2}\sqrt{b^2 + 4h^2}} \right\} = 4 \sin^{-1} \left\{ \frac{lb}{\sqrt{l^2 + 4h^2}\sqrt{b^2 + 4h^2}} \right\} \\ \therefore \omega_{ABCD} &= 4 \sin^{-1} \left\{ \frac{lb}{\sqrt{(l^2 + 4h^2)(b^2 + 4h^2)}} \right\} \quad \dots \dots \dots (5) \end{aligned}$$

Note: This is the standard formula to find out the value of solid angle subtended by a rectangular plane of size $a \times b$ at any point lying at a normal height h from the centre.

- **Rhombus-like Plane**

Let there be a rhombus-like plane ABCD having diagonals $AC = 2d_1$ & $BD = 2d_2$ bisecting each other at right angle at the centre 'O' and a given point say $P(0, 0, h)$ at a height 'h' lying on the normal axis passing through the centre 'O' (i.e. foot of perpendicular) (See the figure 15)

Join all the vertices A, B, C & D to the F.O.P. to divide the rhombus ABCD into elementary right triangles $\Delta AOB, \Delta BOC, \Delta COD$ & ΔAOD . Now, area of rhombus ABCD

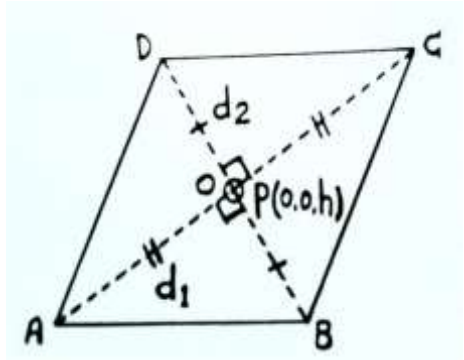


Fig 15: Point P lying on the normal axis passing through the centre of rhombus

$$\begin{aligned} \Rightarrow A_{ABCD} &= A_{\Delta AOB} + A_{\Delta BOC} + A_{\Delta COD} + A_{\Delta AOD} \\ &= 4(A_{\Delta AOB}) \quad \text{By symmetry} \end{aligned}$$

Hence, using **Element Method** by replacing area by corresponding value of solid angle, the solid angle subtended by the rhombus-like plane ABCD at the given point P.

$$\omega_{ABCD} = 4\omega_{\Delta AOB}$$

From eq(3), we know that solid angle subtended by a right triangular plane at any point lying on the vertical passing through right angled vertex is given as

$$\omega = \cos^{-1} \left\{ \frac{h(a^2\sqrt{h^2 + b^2} + b^2\sqrt{h^2 + a^2})}{h^2(a^2 + b^2) + a^2b^2} \right\}$$

On setting, $a = d_1$ & $b = d_2$ in the above equation, we get

$$\omega_{\Delta AOB} = \cos^{-1} \left\{ \frac{h \left(d_1^2 \sqrt{h^2 + d_2^2} + d_2^2 \sqrt{h^2 + d_1^2} \right)}{h^2(d_1^2 + d_2^2) + d_1^2 d_2^2} \right\}$$

Hence, the solid angle subtended by rhombus-like plane ABCD at the given point $P(0, 0, h)$ lying at a normal height h from centre 'O'

$$\Rightarrow \omega_{ABCD} = 4 \cos^{-1} \left\{ \frac{h \left(d_1^2 \sqrt{h^2 + d_2^2} + d_2^2 \sqrt{h^2 + d_1^2} \right)}{h^2 (d_1^2 + d_2^2) + d_1^2 d_2^2} \right\} \dots \dots \dots (6)$$

Note: This is the standard formula to find out the value of solid angle subtended by a rhombus-like plane having diagonals $2d_1$ & $2d_2$ at any point lying at a normal height h from the centre.

- **Regular Polygonal Plane**

Let there be a regular polygonal plane $A_1 A_2 A_3 \dots A_n$ having 'n' no. of the sides each equal to the length ' a ' & a given point say $P(0, 0, h)$ at a height ' h ' lying on the normal height h from the centre 'O'. (i.e. foot of perpendicular) (See the figure 16)

Join all the vertices $A_1, A_2, A_3, \dots, A_n$ to the F.O.P. 'O' to divide the polygon into elementary triangles $\Delta O A_1 A_2, \Delta O A_2 A_3, \dots, \Delta O A_{n-1} A_n$ which are congruent hence the area of polygon

$$A_{reg.poly.} = A_{\Delta O A_1 A_2} + A_{\Delta O A_2 A_3} + A_{\Delta O A_3 A_4} + \dots + A_{\Delta O A_n A_1}$$

$$= n(A_{\Delta O A_1 A_2}) = n(A_{\Delta O M A_1} + A_{\Delta O M A_2})$$

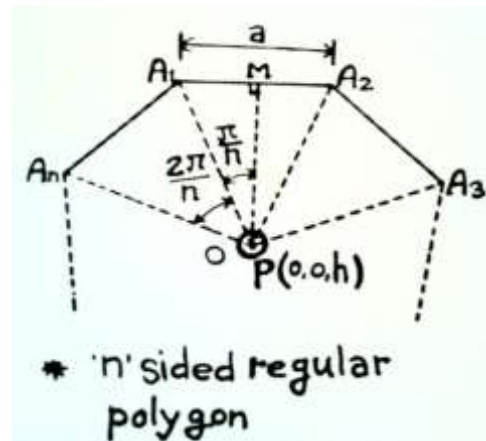


Fig 16: Point P lying on the normal axis passing through the centre of regular polygon

Now, draw a perpendicular OM from F.O.P. 'O' to opposite side $A_1 A_2$ to divide $\Delta O A_1 A_2$ into two sub-elementary right triangles $\Delta O M A_1$ & $\Delta O M A_2$ which are congruent. Hence area of polygon

$$A_{reg.poly.} = n \times 2(A_{\Delta O M A_1}) = 2n(A_{\Delta O M A_1})$$

Hence, using **Element Method** by replacing area by corresponding value of solid angle, the solid angle subtended by the regular polygonal plane $A_1 A_2 A_3 \dots A_n$ at the given point P.

$$\omega_{ABCD} = 2n(\omega_{\Delta O M A_1})$$

Now, using standard formula-1, solid angle subtended by the right $\Delta O M A_1$ at the point $P(0, 0, h)$

$$\omega_{\Delta O M A_1} = \sin^{-1} \left\{ \frac{A_1 M}{\sqrt{(A_1 M)^2 + (OM)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{A_1 M}{\sqrt{(A_1 M)^2 + (OM)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OM)^2}} \right) \right\}$$

Now, setting the corresponding values in above expression as follows

$$A_1 M = \frac{a}{2}, \quad OM = \frac{a}{2} \cot \frac{\pi}{n} \quad \& \quad PO = h, \quad \text{we have}$$

$$\omega_{\Delta O M A_1} = \sin^{-1} \left\{ \frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} \cot \frac{\pi}{n}\right)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} \cot \frac{\pi}{n}\right)^2}} \right) \left(\frac{h}{\sqrt{h^2 + \left(\frac{a}{2} \cot \frac{\pi}{n}\right)^2}} \right) \right\}$$

$$\begin{aligned}
&= \sin^{-1} \left\{ \frac{1}{\operatorname{cosec} \frac{\pi}{n}} \right\} - \sin^{-1} \left\{ \left(\frac{1}{\operatorname{cosec} \frac{\pi}{n}} \right) \left(\frac{2h}{\sqrt{4h^2 + a^2 \cot^2 \frac{\pi}{n}}} \right) \right\} \\
&= \sin^{-1} \left\{ \sin \frac{\pi}{n} \right\} - \sin^{-1} \left\{ \frac{2h \sin \frac{\pi}{n}}{\sqrt{4h^2 + a^2 \cot^2 \frac{\pi}{n}}} \right\} = \frac{\pi}{n} - \sin^{-1} \left\{ \frac{2h \sin \frac{\pi}{n}}{\sqrt{4h^2 + a^2 \cot^2 \frac{\pi}{n}}} \right\}
\end{aligned}$$

Hence, the solid angle subtended by regular polygonal plane $A_1 A_2 A_3 \dots A_n$ at the given point $P(0, 0, h)$ lying at a normal height h from centre 'O'

$$\begin{aligned}
&= 2n \left[\frac{\pi}{n} - \sin^{-1} \left\{ \frac{2h \sin \frac{\pi}{n}}{\sqrt{4h^2 + a^2 \cot^2 \frac{\pi}{n}}} \right\} \right] \\
\Rightarrow \omega_{reg.poly.} &= 2\pi - 2n \sin^{-1} \left\{ \frac{2h \sin \frac{\pi}{n}}{\sqrt{4h^2 + a^2 \cot^2 \frac{\pi}{n}}} \right\} \quad \forall n \geq 3 \quad \dots \dots \dots (7)
\end{aligned}$$

Note: This is the standard formula to find out the value of solid angle subtended by a regular polygonal plane, having n number of sides each of length a , at any point lying at a normal height h from the centre.

• Regular Pentagonal Plane

Let there be a regular pentagonal plane ABCDE having each side of length a and a given point say $P(0, 0, h)$ lying at a normal height 'h' from the centre 'O' (i.e. foot of perpendicular) (See figure 17)

Join all the vertices A, B, C, D & E to the F.O.P. 'O' to divide the pentagon ABCDE into elementary triangles $\triangle ABC, \triangle ACD$ & $\triangle ADE$.

Now, draw perpendiculars AN, AQ & AM to the opposite sides BC, CD & DE in $\triangle ABC, \triangle ACD$ & $\triangle ADE$ respectively to divide each elementary triangle into two right triangles. Hence the area of regular pentagon

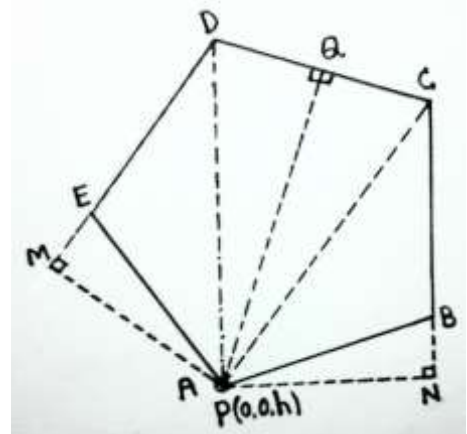


Fig 17: Point P lying on the normal axis passing through the vertex A of regular pentagon

$$A_{reg.penta} = 2(A_{\triangle ABC} + A_{\triangle AQC}) \quad \text{by symmetry}$$

$$= 2(A_{\triangle ANC} - A_{\triangle ANB} + A_{\triangle AQC}) \quad (A_{\triangle ABC} = A_{\triangle ANC} - A_{\triangle ANB})$$

Hence, using **Element Method** by replacing areas by corresponding values of solid angle, the solid angle subtended by the regular pentagonal plane ABCDE at the given point P.

$$\omega_{reg.penta.} = 2(\omega_{\triangle ANC} - \omega_{\triangle ANB} + \omega_{\triangle AQC}) = 2(\omega_{\triangle ANC} + \omega_{\triangle AQC} - \omega_{\triangle ANB}) =$$

Necessary dimensions can be calculated by the figure as follows

$$AN = a \cos 18^\circ, \quad BN = a \sin 18^\circ, \quad CN = a + a \sin 18^\circ \quad \& \quad AQ = \frac{a}{2} \cot 18^\circ$$

Now, using standard formula-1, solid angle subtended by the right ΔANC at the point $P(0, 0, h)$

$$\begin{aligned}
\omega_{\Delta ANC} &= \sin^{-1} \left\{ \frac{CN}{\sqrt{(CN)^2 + (AN)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CN}{\sqrt{(CN)^2 + (AN)^2}} \right) \left(\frac{PA}{\sqrt{(PA)^2 + (AN)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{a + a \sin 18^\circ}{\sqrt{(a + a \sin 18^\circ)^2 + (a \cos 18^\circ)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{a + a \sin 18^\circ}{\sqrt{(a + a \sin 18^\circ)^2 + (a \cos 18^\circ)^2}} \right) \left(\frac{h}{\sqrt{h^2 + (a \cos 18^\circ)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{1 + \sin 18^\circ}{\sqrt{2(1 + \sin 18^\circ)}} \right\} - \sin^{-1} \left\{ \left(\frac{1 + \sin 18^\circ}{\sqrt{2(1 + \sin 18^\circ)}} \right) \left(\frac{h}{\sqrt{h^2 + a^2 \cos^2 18^\circ}} \right) \right\} \\
&= \sin^{-1} \left\{ \sqrt{\frac{1 + \sin 18^\circ}{2}} \right\} - \sin^{-1} \left\{ \left(\sqrt{\frac{1 + \sin 18^\circ}{2}} \right) \left(\frac{h}{\sqrt{h^2 + a^2 \cos^2 18^\circ}} \right) \right\} \\
\omega_{\Delta ANC} &= \frac{3\pi}{10} - \sin^{-1} \left(\frac{h \cos 36^\circ}{\sqrt{h^2 + a^2 \cos^2 18^\circ}} \right) \quad \left(\text{since, } \sqrt{\frac{1 + \sin 18^\circ}{2}} = \cos 36^\circ \right)
\end{aligned}$$

Similarly, we get solid angle subtended by right ΔANB at the given point 'P'

$$\begin{aligned}
\omega_{\Delta ANB} &= \sin^{-1} \left\{ \frac{BN}{\sqrt{(BN)^2 + (AN)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{BN}{\sqrt{(BN)^2 + (AN)^2}} \right) \left(\frac{PA}{\sqrt{(PA)^2 + (AN)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{a \sin 18^\circ}{\sqrt{(a \sin 18^\circ)^2 + (a \cos 18^\circ)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{a \sin 18^\circ}{\sqrt{(a \sin 18^\circ)^2 + (a \cos 18^\circ)^2}} \right) \left(\frac{h}{\sqrt{h^2 + (a \cos 18^\circ)^2}} \right) \right\} \\
&= \sin^{-1} \{ \sin 18^\circ \} - \sin^{-1} \left\{ (\sin 18^\circ) \left(\frac{h}{\sqrt{h^2 + a^2 \cos^2 18^\circ}} \right) \right\} \\
\omega_{\Delta ANB} &= \frac{\pi}{10} - \sin^{-1} \left(\frac{h \sin 18^\circ}{\sqrt{h^2 + a^2 \cos^2 18^\circ}} \right)
\end{aligned}$$

Similarly, we get solid angle subtended by right ΔAQC at the given point 'P'

$$\begin{aligned}
\omega_{\Delta AQC} &= \sin^{-1} \left\{ \frac{CQ}{\sqrt{(CQ)^2 + (AQ)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CQ}{\sqrt{(CQ)^2 + (AQ)^2}} \right) \left(\frac{PA}{\sqrt{(PA)^2 + (AQ)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} \cot 18^\circ\right)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2} \cot 18^\circ\right)^2}} \right) \left(\frac{h}{\sqrt{h^2 + \left(\frac{a}{2} \cot 18^\circ\right)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{1}{\operatorname{cosec} 18^\circ} \right\} - \sin^{-1} \left\{ \left(\frac{1}{\operatorname{cosec} 18^\circ} \right) \left(\frac{2h}{\sqrt{4h^2 + a^2 \cot^2 18^\circ}} \right) \right\} \\
&= \frac{\pi}{10} - \sin^{-1} \left(\frac{2h \sin 18^\circ}{\sqrt{4h^2 + a^2 \cot^2 18^\circ}} \right)
\end{aligned}$$

Hence on setting the corresponding values, solid angle subtended by the regular pentagonal plane ABCDE at the given point P is given as

$$\omega_{\text{reg.penta.}} = 2[\omega_{\Delta ANC} + \omega_{\Delta AQC} - \omega_{\Delta ANB}]$$

$$= 2 \left[\frac{3\pi}{10} - \sin^{-1} \left(\frac{h \cos 36^\circ}{\sqrt{h^2 + a^2 \cos^2 18^\circ}} \right) + \frac{\pi}{10} - \sin^{-1} \left(\frac{2h \sin 18^\circ}{\sqrt{4h^2 + a^2 \cot^2 18^\circ}} \right) - \frac{\pi}{10} + \sin^{-1} \left(\frac{h \sin 18^\circ}{\sqrt{h^2 + a^2 \cos^2 18^\circ}} \right) \right]$$

$$\omega_{reg.penta.} = 2 \left[\frac{3\pi}{10} + \sin^{-1} \left(\frac{h \sin 18^\circ}{\sqrt{h^2 + a^2 \cos^2 18^\circ}} \right) - \sin^{-1} \left(\frac{h \cos 36^\circ}{\sqrt{h^2 + a^2 \cos^2 18^\circ}} \right) - \sin^{-1} \left(\frac{2h \sin 18^\circ}{\sqrt{4h^2 + a^2 \cot^2 18^\circ}} \right) \right]$$

Note: This is the standard formula to find out the value of solid angle subtended by a regular pentagonal plane, having each side of length a , at any point lying at a normal height h from any of the vertices.

• Regular Hexagonal Plane

By following the same procedure as in case of a regular pentagon, we can divide the hexagon into sub-elementary right triangles.

(As shown in the figure 18)

By using Element Method, solid angle subtended by given regular hexagonal plane ABCDEF at the given point P lying at a normal height h from vertex 'A' (i.e. foot of perpendicular)

$$\Rightarrow \omega_{reg.hexa.} = 2(\omega_{\Delta ANC} + \omega_{\Delta ACD} - \omega_{\Delta ANB})$$

Necessary dimensions can be calculated by the figure as follows

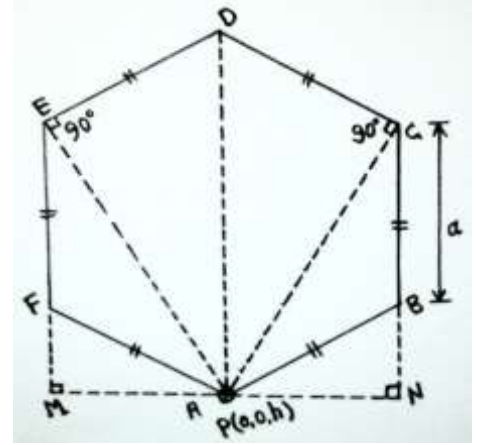


Fig 18: Point P lying on the normal axis passing through the vertex A of regular hexagon

$$AN = a \cos 30^\circ = \frac{a\sqrt{3}}{2}, \quad BN = a \sin 30^\circ = \frac{a}{2}, \quad CN = a + \frac{a}{2} = \frac{3a}{2} \quad \& \quad AC = \frac{3a}{2 \sin 60^\circ} = a\sqrt{3}$$

Now, using standard formula-1, solid angle subtended by the right ΔANC at the point $P(0, 0, h)$

$$\omega_{\Delta ANC} = \sin^{-1} \left\{ \frac{CN}{\sqrt{(CN)^2 + (AN)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CN}{\sqrt{(CN)^2 + (AN)^2}} \right) \left(\frac{PA}{\sqrt{(PA)^2 + (AN)^2}} \right) \right\}$$

$$= \sin^{-1} \left\{ \frac{\frac{3a}{2}}{\sqrt{\left(\frac{3a}{2}\right)^2 + \left(\frac{a\sqrt{3}}{2}\right)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{\frac{3a}{2}}{\sqrt{\left(\frac{3a}{2}\right)^2 + \left(\frac{a\sqrt{3}}{2}\right)^2}} \right) \left(\frac{h}{\sqrt{h^2 + \left(\frac{a\sqrt{3}}{2}\right)^2}} \right) \right\}$$

$$= \sin^{-1} \left\{ \frac{\sqrt{3}}{2} \right\} - \sin^{-1} \left\{ \left(\frac{\sqrt{3}}{2} \right) \left(\frac{2h}{\sqrt{4h^2 + 3a^2}} \right) \right\} = \frac{\pi}{3} - \sin^{-1} \left(\frac{h\sqrt{3}}{\sqrt{4h^2 + 3a^2}} \right)$$

Similarly, we get solid angle subtended by right ΔANB at the given point $P(0, 0, h)$

$$\omega_{\Delta ANB} = \sin^{-1} \left\{ \frac{BN}{\sqrt{(BN)^2 + (AN)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{BN}{\sqrt{(BN)^2 + (AN)^2}} \right) \left(\frac{PA}{\sqrt{(PA)^2 + (AN)^2}} \right) \right\}$$

$$= \sin^{-1} \left\{ \frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a\sqrt{3}}{2}\right)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a\sqrt{3}}{2}\right)^2}} \right) \left(\frac{h}{\sqrt{h^2 + \left(\frac{a\sqrt{3}}{2}\right)^2}} \right) \right\}$$

$$= \sin^{-1} \left\{ \frac{1}{2} \right\} - \sin^{-1} \left\{ \left(\frac{1}{2} \right) \left(\frac{2h}{\sqrt{4h^2 + 3a^2}} \right) \right\} = \frac{\pi}{6} - \sin^{-1} \left(\frac{h}{\sqrt{4h^2 + 3a^2}} \right)$$

Similarly, we get solid angle subtended by right $\triangle ACD$ at the given point $P(0, 0, h)$

$$\begin{aligned} \omega_{\triangle ACD} &= \sin^{-1} \left\{ \frac{CD}{\sqrt{(CD)^2 + (AC)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CD}{\sqrt{(CD)^2 + (AC)^2}} \right) \left(\frac{PA}{\sqrt{(PA)^2 + (AC)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{a}{\sqrt{a^2 + (a\sqrt{3})^2}} \right\} - \sin^{-1} \left\{ \left(\frac{a}{\sqrt{h^2 + (a\sqrt{3})^2}} \right) \left(\frac{h}{\sqrt{h^2 + (a\sqrt{3})^2}} \right) \right\} \\ \omega_{\triangle ACD} &= \sin^{-1} \left\{ \frac{1}{2} \right\} - \sin^{-1} \left\{ \left(\frac{1}{2} \right) \left(\frac{h}{\sqrt{h^2 + 3a^2}} \right) \right\} = \frac{\pi}{6} - \sin^{-1} \left(\frac{h}{2\sqrt{h^2 + 3a^2}} \right) \end{aligned}$$

Hence on setting the corresponding values, solid angle subtended by the regular hexagonal plane ABCDEF at the given point P is given as

$$\begin{aligned} \omega_{reg.hexa.} &= 2[\omega_{\triangle ANC} + \omega_{\triangle ACD} - \omega_{\triangle ANB}] \\ &= 2 \left[\frac{\pi}{3} - \sin^{-1} \left(\frac{h\sqrt{3}}{\sqrt{4h^2 + 3a^2}} \right) + \frac{\pi}{6} - \sin^{-1} \left(\frac{h}{2\sqrt{h^2 + 3a^2}} \right) - \frac{\pi}{6} + \sin^{-1} \left(\frac{h}{\sqrt{4h^2 + 3a^2}} \right) \right] \\ \omega_{reg.hexa.} &= 2 \left[\frac{\pi}{3} + \sin^{-1} \left(\frac{h}{\sqrt{4h^2 + 3a^2}} \right) - \sin^{-1} \left(\frac{h\sqrt{3}}{\sqrt{4h^2 + 3a^2}} \right) - \sin^{-1} \left(\frac{h}{2\sqrt{h^2 + 3a^2}} \right) \right] \end{aligned}$$

Note: This is the standard formula to find out the value of solid angle subtended by a regular hexagonal plane, having each side of length a , at any point lying at a normal height h from any of the vertices.

Thus, all above standard results are obtained by analytical method of HCR's Theory using single **standard formula-1 of right triangular plane** only. It is obvious that this theory can be applied to find out the solid angle subtended by any polygonal plane (i.e. **plane bounded by the straight lines only**) provided the location of foot of perpendicular (**F.O.P.**) is known.

Now, we are interested to calculate solid angle subtended by different polygonal planes at different points in the space by tracing the diagram, specifying the F.O.P. & measuring the necessary dimensions & calculating.

X. Graphical Applications of Theory of Polygon

Graphical Method:

This method is similar to the analytical method which is applicable for some particular configurations of polygon & locations of given point in the space. But graphical method is applicable for any configuration of polygonal plane & location of the point in the space. This is the **method of tracing, measurements & mathematical calculations** which requires the following parameters to be already known

1. Geometrical shape & dimensions of the polygonal plane
2. Normal distance (h) of the given point from the plane of polygon
3. Location of foot of perpendicular (F.O.P.) drawn from given point to the plane of polygon

First let's know the working steps of the graphical method as follows

Step 1: Trace the diagram of the given polygon with the help of known sides & angles.

Step 2: Draw a perpendicular to the plane of polygon & specify the location of F.O.P.

Step 3: Divide the polygon into elementary triangles then each elementary triangle into two sub-elementary right triangles all having common vertex at the F.O.P.

Step 4: Find the area of the polygon as the algebraic sum of areas of sub-elementary right triangles i.e. area of each of the right triangles must be taken with proper sign (positive or negative depending on the area is inside or outside the boundary of polygon)

Step 5: Replace each area of sub-elementary right triangle by the solid angle subtended by that right triangle at the given point in the space.

Step 6: Measure the necessary dimensions (i.e. distances) & set them into standard formula-1 to calculate the solid angle subtended by each of the sub-elementary right triangles.

Step 7: Thus, find out the value of solid angle subtended by given polygonal plane at the given point by taking the algebraic sum of solid angles subtended by the sub-elementary right triangles at the same point in the space.

We are interested to directly apply the above steps without mentioning them in the following numerical examples

Numerical Examples:

Example 1: Let's find out the value of solid angle subtended by a triangular ABC having sides $AB = 8.6\text{cm}$, $BC = 4\text{cm}$ & $AC = 5.5\text{cm}$ at a point P lying at a normal height 3cm from the vertex 'A'.

Sol. Draw the triangle ABC with known values of the sides & specify the location of given point P by $P(0, 0, 3)$ perpendicularly outwards to the plane of paper & F.O.P. (i.e. vertex 'A') (as shown in the figure 19 below)

Divide $\triangle ABC$ into two right triangles $\triangle ANB$ & $\triangle ACN$ by drawing a perpendicular AN to the opposite side BC (extended line). All must have common vertex at F.O.P.

It is clear from the diagram, the solid angle subtended by $\triangle ABC$ at the point 'P' is given by Element Method as follows

$$\omega_{\triangle ABC} = \omega_{\triangle ANB} - \omega_{\triangle ANC}$$

Now, measure the necessary dimensions & perform the following calculations, by using standard formula-1

$$AN = 4.4\text{cm}, \quad CN = 3.6\text{cm} \quad (\text{from the diagram})$$

$$BN = BC + CN = 4 + 3.6 = 7.6\text{cm}$$

Now, solid angle subtended by right $\triangle ANB$ at the given point 'P'

On setting the corresponding values in formula of right triangle

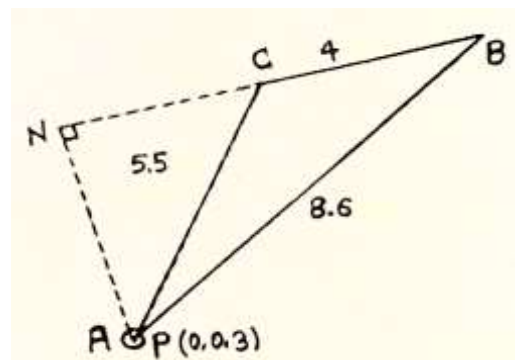


Fig 19: Point P is lying perpendicularly outwards to the plane of paper. All the dimensions are in cm.

$$\begin{aligned}
\Rightarrow \omega_{\Delta ANB} &= \sin^{-1} \left\{ \frac{BN}{\sqrt{(BN)^2 + (AN)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{BN}{\sqrt{(BN)^2 + (AN)^2}} \right) \left(\frac{PA}{\sqrt{(PA)^2 + (AN)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{7.6}{\sqrt{7.6^2 + 4.4^2}} \right\} - \sin^{-1} \left\{ \left(\frac{7.6}{\sqrt{7.6^2 + 4.4^2}} \right) \left(\frac{3}{\sqrt{3^2 + 4.4^2}} \right) \right\} \\
&= 1.046000555 - 0.509254517 = \mathbf{0.536746038 \text{ sr}}
\end{aligned}$$

Similarly, solid angle subtended by right ΔANC at the given point 'P'

$$\begin{aligned}
\Rightarrow \omega_{\Delta ANC} &= \sin^{-1} \left\{ \frac{CN}{\sqrt{(CN)^2 + (AN)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CN}{\sqrt{(CN)^2 + (AN)^2}} \right) \left(\frac{PA}{\sqrt{(PA)^2 + (AN)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{3.6}{\sqrt{3.6^2 + 4.4^2}} \right\} - \sin^{-1} \left\{ \left(\frac{3.6}{\sqrt{3.6^2 + 4.4^2}} \right) \left(\frac{3}{\sqrt{3^2 + 4.4^2}} \right) \right\} \\
&= 0.68572951 - 0.364761147 = \mathbf{0.320968363 \text{ sr}}
\end{aligned}$$

Hence, solid angle subtended by ΔABC at the point 'P' (by Element Method)

$$\Rightarrow \omega_{\Delta ABC} = \omega_{\Delta ANB} - \omega_{\Delta ANC} = 0.536746038 - 0.320968363 = \mathbf{0.215777675 \text{ sr}}$$

Example 2: Let's find out solid angle subtended by a quadrilateral ABCD having sides $AB = 6\text{cm}$, $BC = 8\text{cm}$, $CD = 7\text{cm}$, $AD = 4\text{cm}$ & $\alpha BAD = 110^\circ$ at a point lying at a normal height 2cm from the vertex 'A' & calculate the total luminous flux intercepted by the plane ABCD if a uniform point-source of 1400 lm is located at the point 'P'

Sol. Draw the quadrilateral ABCD with known values of the sides & angle & specify the location of given point P by $P(0, 0, 2)$ perpendicularly outwards to the plane of paper & F.O.P. (i.e. vertex 'A') (as shown in the fig 20)

Divide the quadrilateral ABCD into two triangles ΔABC & ΔADC by joining the vertex C to the F.O.P. 'A'. Further divide ΔABC & ΔADC into two right triangles ΔAMB & ΔAMC and ΔANC & ΔAND respectively simply by drawing perpendicular to the opposite side in ΔABC & ΔADC having common vertex at F.O.P.

It is clear from the diagram, the solid angle subtended by quadrilateral ABCD at the point 'P' is given by Element Method

$$\omega_{ABCD} = \omega_{\Delta ABC} + \omega_{\Delta ACD} = (\omega_{\Delta AMC} + \omega_{\Delta AMB}) + (\omega_{\Delta ANC} - \omega_{\Delta AND})$$

Now, measure the necessary dimensions then perform the following calculations

$$\begin{aligned}
AN &= 3.8\text{cm}, \quad DN = 1.2\text{cm}, \\
AM &= 5.9\text{cm}, \quad (\text{from the diagram})
\end{aligned}$$

$$BM = 1.2\text{cm} \Rightarrow CN = CD + DN = 7 + 1.2 = 8.2\text{cm} \text{ \& }$$

$$CM = BC - BM = 8 - 1.2 = 6.8\text{cm}$$

Now, solid angle subtended by right ΔANC at the given point 'P'

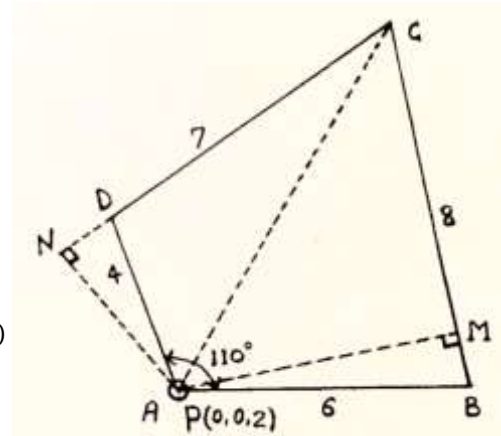


Fig 20: Point P is lying perpendicularly outwards to the plane of paper. All the dimensions are in cm.

On setting the corresponding values in formula of right triangle

$$\begin{aligned}
 \Rightarrow \omega_{\Delta ANC} &= \sin^{-1} \left\{ \frac{CN}{\sqrt{(CN)^2 + (AN)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CN}{\sqrt{(CN)^2 + (AN)^2}} \right) \left(\frac{PA}{\sqrt{(PA)^2 + (AN)^2}} \right) \right\} \\
 &= \sin^{-1} \left\{ \frac{8.2}{\sqrt{8.2^2 + 3.8^2}} \right\} - \sin^{-1} \left\{ \left(\frac{8.2}{\sqrt{8.2^2 + 3.8^2}} \right) \left(\frac{2}{\sqrt{2^2 + 3.8^2}} \right) \right\} \\
 &= 1.136842957 - 0.436286431 = \mathbf{0.700556526 \text{ sr}}
 \end{aligned}$$

Similarly, solid angle subtended by right ΔAND at the given point 'P'

$$\begin{aligned}
 \Rightarrow \omega_{\Delta AND} &= \sin^{-1} \left\{ \frac{DN}{\sqrt{(DN)^2 + (AN)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{DN}{\sqrt{(DN)^2 + (AN)^2}} \right) \left(\frac{PA}{\sqrt{(PA)^2 + (AN)^2}} \right) \right\} \\
 &= \sin^{-1} \left\{ \frac{1.2}{\sqrt{1.2^2 + 3.8^2}} \right\} - \sin^{-1} \left\{ \left(\frac{1.2}{\sqrt{1.2^2 + 3.8^2}} \right) \left(\frac{2}{\sqrt{2^2 + 3.8^2}} \right) \right\} \\
 &= 0.305878871 - 0.140714774 = \mathbf{0.165164097 \text{ sr}}
 \end{aligned}$$

Similarly, solid angle subtended by right ΔAMC at the given point 'P'

$$\begin{aligned}
 \Rightarrow \omega_{\Delta AMC} &= \sin^{-1} \left\{ \frac{CM}{\sqrt{(CM)^2 + (AM)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CM}{\sqrt{(CM)^2 + (AM)^2}} \right) \left(\frac{PA}{\sqrt{(PA)^2 + (AM)^2}} \right) \right\} \\
 &= \sin^{-1} \left\{ \frac{6.8}{\sqrt{6.8^2 + 5.9^2}} \right\} - \sin^{-1} \left\{ \left(\frac{6.8}{\sqrt{6.8^2 + 5.9^2}} \right) \left(\frac{2}{\sqrt{2^2 + 5.9^2}} \right) \right\} \\
 &= 0.856146031 - 0.24492976 = \mathbf{0.611216271 \text{ sr}}
 \end{aligned}$$

Similarly, solid angle subtended by right ΔAMB at the given point 'P'

$$\begin{aligned}
 \Rightarrow \omega_{\Delta AMB} &= \sin^{-1} \left\{ \frac{BM}{\sqrt{(BM)^2 + (AM)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{BM}{\sqrt{(BM)^2 + (AM)^2}} \right) \left(\frac{PA}{\sqrt{(PA)^2 + (AM)^2}} \right) \right\} \\
 &= \sin^{-1} \left\{ \frac{1.2}{\sqrt{1.2^2 + 5.9^2}} \right\} - \sin^{-1} \left\{ \left(\frac{1.2}{\sqrt{1.2^2 + 5.9^2}} \right) \left(\frac{2}{\sqrt{2^2 + 5.9^2}} \right) \right\} \\
 &= 0.200652877 - 0.064029809 = \mathbf{0.136623068 \text{ sr}}
 \end{aligned}$$

Hence, solid angle subtended by *quadrilateral ABCD* at the point 'P' (by Element Method)

$$\begin{aligned}
 \Rightarrow \omega_{ABCD} &= \omega_{\Delta ABC} + \omega_{\Delta ACD} = (\omega_{\Delta AMC} + \omega_{\Delta AMB}) + (\omega_{\Delta ANC} - \omega_{\Delta AND}) \\
 \therefore \omega_{ABCD} &= 0.611216271 + 0.136623068 + 0.700556526 - 0.165164097 = \mathbf{1.283231768 \text{ sr}}
 \end{aligned}$$

Calculation of Luminous Flux: If a **uniform point-source** of 1400 lm is located at the given point 'P' then the **total luminous flux intercepted** by the quadrilateral plane ABCD

$$= \frac{\text{solid angle} \times \text{total flux emitted by source}}{4\pi} = \frac{1.283231768 \times 1400}{4\pi} = \mathbf{142.9628753 \text{ lm (Lumen)}}$$

It means that only 142.9628753 lm out of 1400 lm flux is striking the quadrilateral plane ABCD & rest of the flux is escaping to the surrounding space. **This result can be experimentally verified. (H.C. Rajpoot)**

Example 3: Let's find out solid angle subtended by a pentagonal plane ABCDE having sides $AB = 6\text{cm}$, $BC = 5.7\text{cm}$, $CD = 6.65\text{cm}$, $DE = 5.5\text{cm}$, $AE = 7.8\text{cm}$, $\alpha BAE = 120^\circ$ & $\alpha ABC = 80^\circ$ at a point 'P' lying at a normal height 6cm from a point 'O' internally dividing the side AB such that $OA:OB = 2:1$ & calculate the total luminous flux intercepted by the plane ABCDE if a uniform point-source of 1400 lm is located at the point 'P'

Sol: Draw the pentagon ABCDE with known values of the sides & angles & specify the location of given point P by $P(0,0,6)$ perpendicularly outwards to the plane of paper & F.O.P. 'O' (as shown in the figure 21)

Divide the pentagon ABCDE into elementary triangles $\triangle OBC$, $\triangle OCD$, $\triangle ODE$ & $\triangle OEA$ by joining all the vertices of pentagon ABCDE to the F.O.P. 'O'. Further divide each of the triangles $\triangle OBC$, $\triangle OCD$, $\triangle ODE$ & $\triangle OEA$ in two right triangles simply by drawing a perpendicular to the opposite side in the respective triangle. (See the diagram)

It is clear from the diagram, the solid angle subtended by pentagonal plane ABCDE at the point 'P' is given by Element Method as follows

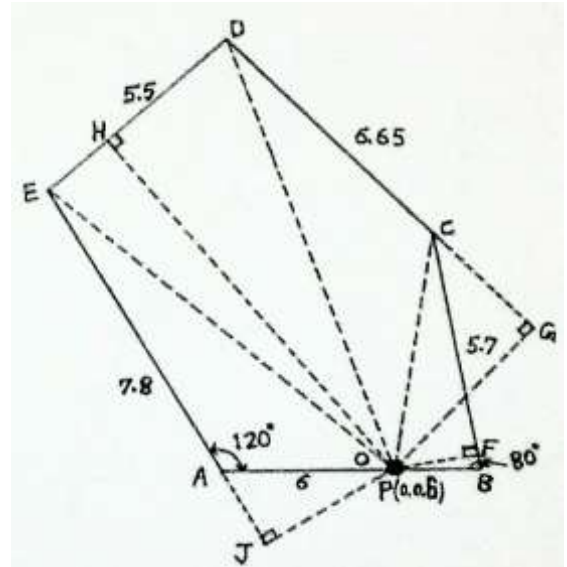


Fig 21: Point P is lying perpendicularly outwards to the plane of paper. All the dimensions are in cm.

$$\omega_{ABCDE} = \omega_{\triangle OBC} + \omega_{\triangle OCD} + \omega_{\triangle ODE} + \omega_{\triangle OEA} \dots \dots (I)$$

From the diagram, it's obvious that the solid angle ω_{ABCDE} subtended by the pentagon ABCDE is expressed as the algebraic sum of solid angles of sub-elementary right triangles only as follows

$$\begin{aligned} \omega_{\triangle OBC} &= \omega_{\triangle OFB} + \omega_{\triangle OFC} & \omega_{\triangle OCD} &= \omega_{\triangle OGD} - \omega_{\triangle OGC} \\ \omega_{\triangle ODE} &= \omega_{\triangle OHD} + \omega_{\triangle OHE} & \omega_{\triangle OEA} &= \omega_{\triangle OJE} - \omega_{\triangle OJA} \end{aligned}$$

Now, setting the values in eq(I), we get

$$\omega_{ABCDE} = (\omega_{\triangle OFB} + \omega_{\triangle OFC}) + (\omega_{\triangle OGD} - \omega_{\triangle OGC}) + (\omega_{\triangle OHD} + \omega_{\triangle OHE}) + (\omega_{\triangle OJE} - \omega_{\triangle OJA}) \dots \dots (II)$$

Now, measure the necessary dimensions & set them into standard formula-1 to find out above values of solid angle subtended by the sub-elementary right triangles at the given point 'P' as follows

$$\begin{aligned} \Rightarrow \omega_{\triangle OFB} &= \sin^{-1} \left\{ \frac{FB}{\sqrt{(FB)^2 + (OF)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{FB}{\sqrt{(FB)^2 + (OF)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OF)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{0.35}{\sqrt{(0.35)^2 + (1.95)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{0.35}{\sqrt{(0.35)^2 + (1.95)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (1.95)^2}} \right) \right\} \\ &= 0.177596167 - 0.168814218 = \mathbf{0.008781949 \text{ sr}} \\ \Rightarrow \omega_{\triangle OFC} &= \sin^{-1} \left\{ \frac{FC}{\sqrt{(FC)^2 + (OF)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{FC}{\sqrt{(FC)^2 + (OF)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OF)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{5.35}{\sqrt{(5.35)^2 + (1.95)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{5.35}{\sqrt{(5.35)^2 + (1.95)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (1.95)^2}} \right) \right\} \\ &= 1.221275136 - 1.105149872 = \mathbf{0.116125264 \text{ sr}} \end{aligned}$$

$$\begin{aligned}
\Rightarrow \omega_{\Delta OGD} &= \sin^{-1} \left\{ \frac{DG}{\sqrt{(DG)^2 + (OG)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{DG}{\sqrt{(DG)^2 + (OG)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OG)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{9.9}{\sqrt{(9.9)^2 + (4.6)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{9.9}{\sqrt{(9.9)^2 + (4.6)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (4.6)^2}} \right) \right\} \\
&= 1.135829376 - 0.803382922 = \mathbf{0.332446454 \text{ sr}} \\
\Rightarrow \omega_{\Delta OGC} &= \sin^{-1} \left\{ \frac{CG}{\sqrt{(CG)^2 + (OG)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CG}{\sqrt{(CG)^2 + (OG)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OG)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{3.25}{\sqrt{(3.25)^2 + (4.6)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{3.25}{\sqrt{(3.25)^2 + (4.6)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (4.6)^2}} \right) \right\} \\
&= 0.615089573 - 0.475672072 = \mathbf{0.139417501 \text{ sr}} \\
\Rightarrow \omega_{\Delta OHD} &= \sin^{-1} \left\{ \frac{DH}{\sqrt{(DH)^2 + (OH)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{DH}{\sqrt{(DH)^2 + (OH)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OH)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{3.7}{\sqrt{(3.7)^2 + (10.25)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{3.7}{\sqrt{(3.7)^2 + (10.25)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (10.25)^2}} \right) \right\} \\
&= 0.346418989 - 0.172376751 = \mathbf{0.174042238 \text{ sr}} \\
\Rightarrow \omega_{\Delta OHE} &= \sin^{-1} \left\{ \frac{EH}{\sqrt{(EH)^2 + (OH)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{EH}{\sqrt{(EH)^2 + (OH)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OH)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{1.8}{\sqrt{(1.8)^2 + (10.25)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{1.8}{\sqrt{(1.8)^2 + (10.25)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (10.25)^2}} \right) \right\} \\
&= 0.173837242 - 0.087488889 = \mathbf{0.086348353 \text{ sr}} \\
\Rightarrow \omega_{\Delta OJE} &= \sin^{-1} \left\{ \frac{EJ}{\sqrt{(EJ)^2 + (OJ)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{EJ}{\sqrt{(EJ)^2 + (OJ)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OJ)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{9.8}{\sqrt{(9.8)^2 + (3.45)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{9.8}{\sqrt{(9.8)^2 + (3.45)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (3.45)^2}} \right) \right\} \\
&= 1.232304566 - 0.957430258 = \mathbf{0.274874308 \text{ sr}} \\
\Rightarrow \omega_{\Delta OJA} &= \sin^{-1} \left\{ \frac{AJ}{\sqrt{(AJ)^2 + (OJ)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{AJ}{\sqrt{(AJ)^2 + (OJ)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OJ)^2}} \right) \right\} \\
&= \sin^{-1} \left\{ \frac{2}{\sqrt{(2)^2 + (3.45)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{2}{\sqrt{(2)^2 + (3.45)^2}} \right) \left(\frac{6}{\sqrt{(6)^2 + (3.45)^2}} \right) \right\} \\
&= 0.525366873 - 0.449793847 = \mathbf{0.075573026 \text{ sr}}
\end{aligned}$$

Hence, by setting the corresponding values in eq(II), solid angle subtended by the pentagonal plane at the given point 'P' is calculated as follows

$$\omega_{ABCDE} = (\omega_{\Delta OFB} + \omega_{\Delta OFC}) + (\omega_{\Delta OGD} - \omega_{\Delta OGC}) + (\omega_{\Delta OHD} + \omega_{\Delta OHE}) + (\omega_{\Delta OJE} - \omega_{\Delta OJA})$$

$$= (0.008781949 + 0.116125264) + (0.332446454 - 0.139417501) + (0.174042238 + 0.086348353) + (0.274874308 - 0.075573026) = \mathbf{0.777628039 \text{ sr}}$$

Calculation of Luminous Flux: If a **uniform point-source** of 1400 lm is located at the given point 'P' then the **total luminous flux intercepted** by the pentagonal plane ABCDE

$$= \frac{\text{solid angle} \times \text{total flux emitted by source}}{4\pi} = \frac{0.777628039 \times 1400}{4\pi} = \mathbf{86.63434241 \text{ lm (Lumen)}}$$

It means that only **86.63434241 lm** out of **1400 lm** flux is striking the pentagonal plane ABCDE & rest of the flux is escaping to the surrounding space. **This result can be experimentally verified. (H.C. Rajpoot)**

Example 4: Let's find out solid angle subtended by a quadrilateral plane ABCD having sides $AB = 8\text{cm}$, $BC = 9\text{cm}$, $CD = 6\text{cm}$, $AD = 4\text{cm}$ & $\angle BAD = 70^\circ$ at a point 'P' lying at a normal height 4cm from a point 'O' outside the quadrilateral ABCD such that $OB = 8\text{cm}$ & $OC = 4.2\text{cm}$ & calculate the total luminous flux intercepted by the plane ABCD if a uniform point-source of 1400 lm is located at the point 'P'

Sol: Draw the quadrilateral ABCD with known values of the sides & angle & specify the location of given point P by $P(0, 0, 4)$ perpendicularly outwards to the plane of paper & F.O.P. 'O' (See the figure 22)

Divide quadrilateral ABCD into elementary triangles $\triangle OAB$, $\triangle ODA$ & $\triangle OCD$ by joining all the vertices of quadrilateral ABCD to the F.O.P. 'O'. Further divide each of the triangles $\triangle OAB$, $\triangle ODA$ & $\triangle OCD$ in two right triangles simply by drawing perpendiculars OE, OG & OF to the opposite sides AB, AD & CD in the respective triangles. (See the diagram)

It is clear from the diagram, the solid angle subtended by quadrilateral plane ABCD at the given point 'P' is given by **Element Method** as follows

Area of quadrilateral ABCD = algebraic sum of areas of elementary triangles

$$\therefore A_{ABCD} = (A_{\triangle OAB} - A_{\triangle OKB}) + (A_{\triangle ODA} - A_{\triangle OJK}) + (A_{\triangle OCD} - A_{\triangle OCJ})$$

Now, replacing areas by corresponding values of solid angle, we get

$$\omega_{ABCD} = (\omega_{\triangle OAB} - \omega_{\triangle OKB}) + (\omega_{\triangle ODA} - \omega_{\triangle OJK}) + (\omega_{\triangle OCD} - \omega_{\triangle OCJ}) \dots \dots \dots (I)$$

Now, draw a perpendicular OH from F.O.P. to the side BC to divide $\triangle OKB$, $\triangle OJK$ & $\triangle OCJ$ into right triangles & express the above values of solid angle as the algebraic sum of solid angles subtended by the right triangles only as follows

$$\begin{array}{lll} \omega_{\triangle OAB} = \omega_{\triangle OEA} - \omega_{\triangle OEB} & \omega_{\triangle ODA} = \omega_{\triangle OGA} - \omega_{\triangle OGD} & \omega_{\triangle OCD} = \omega_{\triangle OFD} - \omega_{\triangle OFC} \\ \omega_{\triangle OKB} = \omega_{\triangle OHB} - \omega_{\triangle OHK} & \omega_{\triangle OJK} = \omega_{\triangle OHK} - \omega_{\triangle OHJ} & \omega_{\triangle OCJ} = \omega_{\triangle OHC} + \omega_{\triangle OHJ} \end{array}$$

Now, setting the above values in eq(I), we get

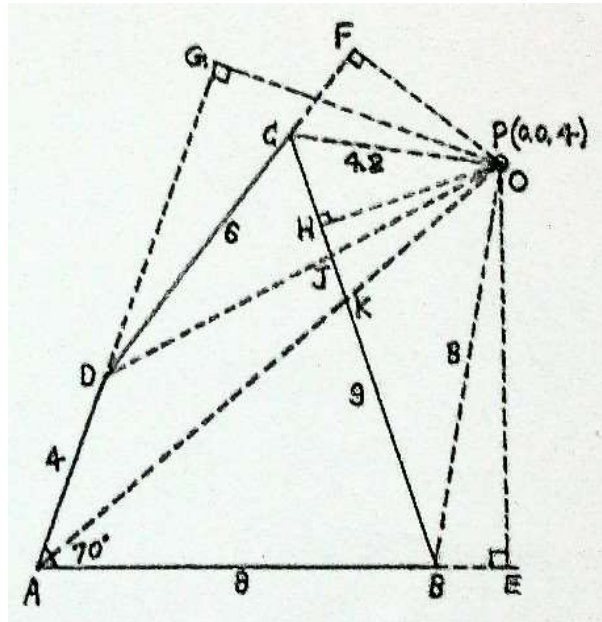


Fig 22: Point P is lying perpendicularly outwards to the plane of paper. All the dimensions are in cm.

$$\omega_{ABCD} = (\omega_{\Delta OEA} - \omega_{\Delta OEB} - \omega_{\Delta OHB} + \omega_{\Delta OHK}) + (\omega_{\Delta OGA} - \omega_{\Delta OGD} - \omega_{\Delta OHK} + \omega_{\Delta OHJ}) + (\omega_{\Delta OFD} - \omega_{\Delta OFC} - \omega_{\Delta OHC} - \omega_{\Delta OHJ})$$

$$\omega_{ABCD} = \omega_{\Delta OEA} + \omega_{\Delta OGA} + \omega_{\Delta OFD} - \omega_{\Delta OEB} - \omega_{\Delta OHB} - \omega_{\Delta OGD} - \omega_{\Delta OFC} - \omega_{\Delta OHC} \quad \dots \dots \dots (II)$$

Now, measure the necessary dimensions & set them into standard formula-1 to find out above values of solid angle subtended by the sub-elementary right triangles at the given point 'P' as follows

$$\begin{aligned} \Rightarrow \omega_{\Delta OEA} &= \sin^{-1} \left\{ \frac{AE}{\sqrt{(AE)^2 + (OE)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{AE}{\sqrt{(AE)^2 + (OE)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OE)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{9.4}{\sqrt{(9.4)^2 + (7.9)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{9.4}{\sqrt{(9.4)^2 + (7.9)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (7.9)^2}} \right) \right\} \\ &= 0.871887063 - 0.353107934 = \mathbf{0.518779129 \text{ sr}} \end{aligned}$$

$$\begin{aligned} \Rightarrow \omega_{\Delta OGA} &= \sin^{-1} \left\{ \frac{AG}{\sqrt{(AG)^2 + (OG)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{AG}{\sqrt{(AG)^2 + (OG)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OG)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{10.5}{\sqrt{(10.5)^2 + (6)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{10.5}{\sqrt{(10.5)^2 + (6)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (6)^2}} \right) \right\} \\ &= 1.051650213 - 0.502496173 = \mathbf{0.54915404 \text{ sr}} \end{aligned}$$

$$\begin{aligned} \Rightarrow \omega_{\Delta OFD} &= \sin^{-1} \left\{ \frac{DF}{\sqrt{(DF)^2 + (OF)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{DF}{\sqrt{(DF)^2 + (OF)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OF)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{8.1}{\sqrt{(8.1)^2 + (3.65)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{8.1}{\sqrt{(8.1)^2 + (3.65)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (3.65)^2}} \right) \right\} \\ &= 1.147429185 - 0.738889475 = \mathbf{0.408539709 \text{ sr}} \end{aligned}$$

$$\begin{aligned} \Rightarrow \omega_{\Delta OEB} &= \sin^{-1} \left\{ \frac{BE}{\sqrt{(BE)^2 + (OE)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{BE}{\sqrt{(BE)^2 + (OE)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OE)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{1.4}{\sqrt{(1.4)^2 + (7.9)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{1.4}{\sqrt{(1.4)^2 + (7.9)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (7.9)^2}} \right) \right\} \\ &= 0.17539422 - 0.078906232 = \mathbf{0.096487988 \text{ sr}} \end{aligned}$$

$$\begin{aligned} \Rightarrow \omega_{\Delta OHB} &= \sin^{-1} \left\{ \frac{BH}{\sqrt{(BH)^2 + (OH)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{BH}{\sqrt{(BH)^2 + (OH)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OH)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{7.1}{\sqrt{(7.1)^2 + (3.8)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{7.1}{\sqrt{(7.1)^2 + (3.8)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (3.8)^2}} \right) \right\} \\ &= 1.079378081 - 0.69346571 = \mathbf{0.385912371 \text{ sr}} \end{aligned}$$

$$\begin{aligned} \Rightarrow \omega_{\Delta OGD} &= \sin^{-1} \left\{ \frac{DG}{\sqrt{(DG)^2 + (OG)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{DG}{\sqrt{(DG)^2 + (OG)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OG)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{6.5}{\sqrt{(6.5)^2 + (6)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{6.5}{\sqrt{(6.5)^2 + (6)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (6)^2}} \right) \right\} \end{aligned}$$

$$= 0.82537685 - 0.419819479 = \mathbf{0.405557371 \text{ sr}}$$

$$\begin{aligned}\Rightarrow \omega_{\Delta OFC} &= \sin^{-1} \left\{ \frac{CF}{\sqrt{(CF)^2 + (OF)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CF}{\sqrt{(CF)^2 + (OF)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OF)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{2.1}{\sqrt{(2.1)^2 + (3.65)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{2.1}{\sqrt{(2.1)^2 + (3.65)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (3.65)^2}} \right) \right\} \\ &= 0.522091613 - 0.37726383 = \mathbf{0.144827783 \text{ sr}}\end{aligned}$$

$$\begin{aligned}\Rightarrow \omega_{\Delta OHC} &= \sin^{-1} \left\{ \frac{CH}{\sqrt{(CH)^2 + (OH)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{CH}{\sqrt{(CH)^2 + (OH)^2}} \right) \left(\frac{PO}{\sqrt{(PO)^2 + (OH)^2}} \right) \right\} \\ &= \sin^{-1} \left\{ \frac{1.9}{\sqrt{(1.9)^2 + (3.8)^2}} \right\} - \sin^{-1} \left\{ \left(\frac{1.9}{\sqrt{(1.9)^2 + (3.8)^2}} \right) \left(\frac{4}{\sqrt{(4)^2 + (3.8)^2}} \right) \right\} \\ &= 0.463647609 - 0.330197223 = \mathbf{0.133450386 \text{ sr}}\end{aligned}$$

Hence, by setting the corresponding values in eq(II), solid angle subtended by the pentagonal plane at the given point 'P' is calculated as follows

$$\begin{aligned}\omega_{ABCD} &= \omega_{\Delta OEA} + \omega_{\Delta OGA} + \omega_{\Delta OFD} - \omega_{\Delta OEB} - \omega_{\Delta OHB} - \omega_{\Delta OGD} - \omega_{\Delta OFC} - \omega_{\Delta OHC} \\ &= 0.518779129 + 0.54915404 + 0.408539709 - 0.096487988 - 0.385912371 - 0.405557371 \\ &\quad - 0.144827783 - 0.133450386 = \mathbf{0.310236979 \text{ sr}}\end{aligned}$$

Calculation of Luminous Flux: If a **uniform point-source** of 1400 lm is located at the given point 'P' then the **total luminous flux intercepted** by the quadrilateral plane ABCD

$$= \frac{\text{solid angle} \times \text{total flux emitted by source}}{4\pi} = \frac{0.310236979 \times 1400}{4\pi} = \mathbf{34.56302412 \text{ lm (Lumen)}}$$

It means that only **34.56302412 lm** out of **1400 lm** flux is striking the quadrilateral plane ABCD & rest of the flux is escaping to the surrounding space. **This result can be experimentally verified.**

Thus, all the mathematical results obtained above can be verified by the experimental results. Although, there had not been any unifying principle to be applied on any polygonal plane for any configuration & location of the point in the space. The symbols & names used above are arbitrary given by the author Mr H.C. Rajpoot.

XI. Conclusion

It is obvious from results obtained above that this theory is a **Unifying Principle** which is easy to apply for any configuration of a given polygon & any location of a point (i.e. observer) in the space by using a simple & systematic procedure & a standard formula. Necessary dimensions can be measured by analytical method or by tracing the diagram of polygon & specifying the location of F.O.P.

Though, it is a little lengthy for random configuration of polygon & location of observer still it can be applied to find the solid angle subtended by polygon in the easier way as compared to any other methods existing so far in the field of 3-D Geometry. Theory of Polygon can be concluded as follows

Applicability: It is easily applied to find out the solid angle subtended by any polygonal plane (i.e. **plane bounded by the straight lines only**) at any point (i.e. observer) in 3D space.

Conditions of Application: This theory is applicable for any polygonal plane & any point in the space if the following parameters are already known

1. Geometrical shape & dimensions of the polygonal plane
2. Normal distance (h) of the given point from the plane of polygon
3. Location of foot of perpendicular (F.O.P.) drawn from given point to the plane of polygon

While, the necessary dimensions (values) used in **master formula-1** (as derived above) are calculated either by analytical method or by graphical method i.e. by tracing the diagram & measuring the dimensions depending on which is easier. Analytical method is limited for some particular location of the point while Graphical method is applicable for all the locations & configuration of polygon w.r.t. the observer in the space. This method can never fail but a little complexity may be there in case of random locations & polygon with higher number of sides.

Steps to be followed:

1. Trace the diagram of the given polygon with the help of known sides & angles.
2. Draw a perpendicular to the plane of polygon & specify the location of F.O.P.
3. Divide the polygon into right triangles having common vertex at the F.O.P. & find the solid angle subtended by the polygon as the algebraic sum of solid angles subtended by right triangles such that algebraic sum of areas of right triangles is equal to the area of polygon.
4. Measure the necessary dimensions & set them into standard formula-1 to calculate solid angle subtended by each of the right triangles & hence solid angle subtended by the polygon at the given point.

Ultimate aim is to find out solid angle, subtended by a polygon at a given point, as the algebraic sum of solid angles subtended by the right triangles, measuring the dimensions, applying **Master/standard formula-1** on each of the right triangle & calculating the required result.

Future Scope: This theory can be easily applied for finding out the solid angle subtended by **3-D objects** which have **surface bounded by the planes only** Ex. **Cube, Cuboid, Prism, Pyramid, Tetrahedron** etc. in 3-D modelling & analysis by tracing the profile of surface of the solid as a polygon in 2-D & specifying the location of a given point & F.O.P. in the plane of profile-polygon as the projection of such solids in 2-D is always a polygon for any configuration of surface of solid in 3D space.

Note: This Theory had been proposed by the author **Mr Harish Chandra Rajpoot (B Tech, ME)**

M.M.M. University of Technology Gorakhpur-273010 (UP) India

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